1. **Stable atom isotopes**

Let’s analyze the 20 lightest element from hydrogen to calcium.

a. Identify all stable isotopes and list their atomic number Z and neutron number N=mass number A - Z.

b. What is the fraction of isotopes with even Z? And the fraction with even N?

c. What caused the different abundance of isotopes with even and odd Z/N? Provide a citation.

d. If an atom is composed of even fermions (electrons+protons+neutrons), it is a boson. Otherwise, a fermion. Is it more likely that an atom is boson or fermion?

e. Can you identify other patterns or rules of isotope stability?

2. **Fine structure an atom**

Fine structure of an atom comes mostly from the coupling between the orbital angular momentum of the electrons and their spins. The interaction Hamiltonian can be approximated by

\[ V_F = \gamma \vec{L} \cdot \vec{S}, \]

where \(\vec{L}\) is the orbital angular momentum, \(\vec{S}\) is the spin and \(\vec{J} = \vec{L} + \vec{S}\) is the total angular momentum, and the constant \(\gamma\) characterizes the strength of the fine structure interactions.

a. Determine the fine structure energy of an electronic orbital \(^{2S+1}L_J\) (Hint: Diagonalize the Hamiltonian given S, L and J.)

b. For alkali atoms in the first excited state \(ns^2P_J\) with \(L = 1\) and \(S = 1/2\), determine the energy splitting between the levels \(J = 3/2\) and \(J = 1/2\) in terms of \(\gamma\).

c. Look up online the fine structure splitting of H, Li, Na, K, Rb, Cs and Fr in their first excited \(L = 1\) state. Test textbook’s claim that the fine structure coupling \(\gamma\) scales with the atomic number \(Z\) as \(\gamma \propto Z^2\). (Hint: you can find the data here: https://steck.us/alkalidata/)

d. D1 and D2 lines of alkali-like atoms refers to the optical transitions between the ground state with \(L = 0\) and the first excited states with \(L = 1\) and \(J = \frac{1}{2}, \frac{3}{2}\) respectively. As an example, the D1 and D2 line wavelengths are 795 nm and 780 nm for Rubidium. Predict the D1 and D2 line wavelengths of Ununennium \(^{119}Uue\).

3. **Hyperfine structure an atom**

Hyperfine structure of an atom comes from the coupling between the total angular momentum of the electrons \(\vec{J}\) and the nuclear spin \(\vec{I}\). The hyperfine interaction Hamiltonian is very well approximated by

\[ V_{HF} = \eta \vec{J} \cdot \vec{I}, \]
which has the same form as the fine structure interactions. The total angular momentum is \( \vec{F} = \vec{J} + \vec{I} = \vec{L} + \vec{S} + \vec{I} \).

Dysprosium has a complex hyperfine structure, shown below. Determine \( J, I \) and \( \eta \) for the two isotopes. [ ]

4. Atom in a magnetic field: Breit-Rabi formula
In the presence of an external magnetic field \( \vec{B} \), the Hamiltonian of an alkali atom in the ground state with \( L = 0, S = \frac{1}{2} \) and nuclear spin \( I \) is given by

\[
H = \gamma \vec{S} \cdot \vec{I} - \vec{\mu}_S \cdot \vec{B} - \vec{\mu}_I \cdot \vec{B},
\]

where \( \vec{\mu}_S = -g_s \mu_B \vec{S}/\hbar \) is the magnetic moment of the electron, \( \vec{\mu}_I = g_I \mu_N \vec{I}/\hbar \) is the magnetic moment of the nucleus, \( g_s \) are the g-factors, \( \mu_B \) and \( \mu_N \) are the Bohr and nucleus magneton.

Here we will show that the eigen-energies are described by the Breit-Rabi formula

\[
E = -\frac{E_{HF}}{2(2l+1)} + \frac{1}{2} \sqrt{1 + \frac{4m_F}{2l+1} x + x^2} - g_I \mu_N m_F B,
\]

where \( E_{HF} = (I + \frac{1}{2}) \gamma \hbar^2 \) is the hyperfine splitting at zero magnetic field, \( m_F = m_I + \frac{1}{2} \), and \( x = \frac{g_s \mu_B + g_I \mu_N}{E_{HF}} \).

a. Writing \( \vec{S} \cdot \vec{I} = \frac{1}{2} \left( S_+ I_- + S_- I_+ \right) + S_z I_z \), show that the Hamiltonian only couples \( |m_I, m_S = \frac{1}{2} > \) and \( |m_I + 1, m_S = -\frac{1}{2} > \). In this sub-space, the Hamiltonian can be written as

\[
H = \begin{pmatrix}
\frac{E_{HF}}{2l+1} m_I + \frac{g_s \mu_B B}{2} - g_I \mu_N m_I B & \frac{E_{HF}}{2l+1} \sqrt{l(l+1) - m_I(m_I + 1)} \\
\frac{E_{HF}}{2l+1} \sqrt{l(l+1) - m_I(m_I + 1)} & \frac{E_{HF}}{2l+1} \left( m_I + 1 \right) - \frac{g_s \mu_B B}{2} - g_I \mu_N (m_I + 1) B
\end{pmatrix}
\]

b. Diagonalize the matrix and show that you get the Breit-Rabi formula.

c. There are two states in the basis of \( |m_I, m_S > \) that do not mix with other states and scale linearly with the magnetic field. Show that in these two states, the electron and nucleus spins are aligned with the field.