Physics 143A: Honors Waves, Optics, and Heat
Spring Quarter 2025
Problem Set #6
Due: 11:59 pm, Tuesday, May 6. Please submit to Canvas.

1. **(Math) Vector calculus (20 points each)**
A scalar field $\phi(\vec{x})$ assigns a single numerical value, called a scalar, to each point in space $\vec{x} = (x, y, z)$, while a vector field $\vec{A}(x, y, z)$ assigns a vector, includes magnitude and direction, to each point in space $\vec{x} = (x, y, z)$. There are 3 common types of vector calculus operations:

Gradient $\nabla\phi \equiv \sum_i \vec{e}_i \partial_i \phi$ returns a vector of greatest increase of ϕ

Divergence $\nabla \cdot \vec{A} \equiv \sum_i \partial_i A_i$ quantifies the amount of source flow

Curl $\nabla \times \vec{A} \equiv \sum_{ijk} \epsilon_{ijk} \hat{e}_i \partial_j A_k = \begin{vmatrix} e_x & e_y & e_z \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$ evaluates the circulation of the field

(Levi-Civita symbol: $\epsilon_{123} = \epsilon_{231} = \epsilon_{312}$ =1, $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$, all others are 0.)

Laplacian $\nabla^2\phi = \nabla \cdot \nabla\phi \equiv \sum_i \partial_i^2 \phi$
$\nabla^2\vec{A} \equiv \sum_{ij} \hat{e}_j \partial_i^2 A_j$ evaluates the curvature of the scalar/vector field

Please practice with the following exercises

a) A vector field carries a source if $\nabla \cdot \vec{A} > 0$ and a sink if $\nabla \cdot \vec{A} < 0$. It carries a circulation if $\nabla \times \vec{A}$ is a non-zero vector. Determine if the following fields carry a source/sink or circulation?
$\vec{A}_1 = (y^2 x, 0, -\mathrm{yx}^2)$
$\vec{A}_2 = (\cos z, \sin x - \sin y, -\cos z)$
$\vec{A}_3 = (yz, \mathrm{xz}, \mathrm{xy})$

b) Prove the following vector identities (ϕ, ψ are scalar fields, $\vec{A}$ is a vector field.)
$\nabla \times \nabla\psi = \nabla \cdot (\nabla \times \vec{A}) = 0$ Note: Gradient field has no circulation and vice versa.
$\nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$ Note: Important identity in QM.
$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$ Note: This is the identify we used to derive EM waves.

2. **Transmission and reflection of waves by an interface (10 points)**
Consider a wave, described by the following equation, in moving toward an interface at $x = 0$,
$$\partial_t^2 \psi(x,t) = v(x)^2 \partial_x^2 \psi(x,t),$$
where the wave propagation velocity changes across the interface
$$v(x < 0) = v_L = \sqrt{\kappa_L/\rho_L}$$
$$v(x > 0) = v_R = \sqrt{\kappa_R/\rho_R},$$
where κ_L (κ_R) are the bulk modulus (=1/compressibility) of the left (right) medium, ρ_L (ρ_R) is the density of the left (right) medium.

Now consider an incident wave coming from the right side toward the interface

$$\psi_{in}(x > 0, t) = A \cos k(x + v_R t).$$

We may solve the equation with the following ansatz

$$\psi(x > 0, t) = Ae^{ik(x+v_R t)} + Be^{ik(-x+v_R t)}$$
$$\psi(x < 0, t) = Ce^{ik_L(x+v_L t)},$$

where A, B and C are the incident, reflected and transmitted wave amplitudes, k_L is the wavenumber of the transmitted wave. Note that the solution is the real part of the ansatz.

A. Show that at the interface we should have the following boundary conditions
First of all the wave is continuous at all times, show that this leads to
Boundary condition 1: $\psi(0^-, t) = \psi(0^+, t)$.
Second, net force on the interface is zero (why?), show that this leads to
Boundary condition 2: $\kappa_L \partial_x \psi(0^-, t) = \kappa_R \partial_x \psi(0^+, t)$.

B. Apply the boundary conditions and show that they lead to
Boundary condition 1 $\Rightarrow A + B = C,\ kv_R = k_L v_L \rightarrow k_L = k\frac{v_R}{v_L}$.
Boundary condition 2 $\Rightarrow \kappa_R kA - \kappa_R kB = \kappa_L k_L C$

C. Given the incident wave amplitude *A* and wavenumber *k*, determine the full solution across the interface.

3. Decibel scale of the strength of sound (10 points each)

Alexander Bell, who invented telephone, introduced the unit of bel, which later became decibel in acoustics: Zero decibel (0 dB) is defined as strength of the sound wave that produces a modulation of pressure $\pm 20\mu Pa$ in the air ($1\mu Pa = 10^{-6} Pa$), which is also the typical limit of human hearing. Decibel is calculated in log scale such that every +20 dB corresponds to 10 times higher pressure. For instance, 20 dB corresponds to $200\mu Pa$ and 40 dB corresponds to $2mPa$ and so on. Human eardrum hearing is damaged above 100dB, corresponding to 2 Pa.

A. How much is the maximum displacement $\psi(x)$ of air molecules in the presence of acoustic waves at human's limits of 0dB and 100dB at frequency $\frac{\omega}{2\pi} = 100$Hz?
(Hint: you may assume the Air pressure is 1 atm=101325 Pa, temperature = 300K and sound velocity is $\sqrt{\frac{k_B T}{m}}$ ≈343m/s, where m is the mean air molecule mass.)

B. A. How much is the variation of air pressure in the presence of acoustic waves at human's limits of 0dB and 100dB at frequency $\frac{\omega}{2\pi} = 100$Hz?

C. Show that sound in the air in principle cannot be louder than 200dB .