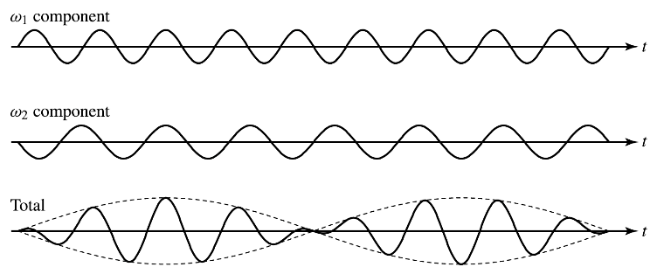
Physics 143A: Honors Waves, Optics, and Heat

Spring Quarter 2025
Problem Set #7
Due: 11:59 pm, Wednesday, May 13. Please submit to Canvas.

1. Propagation of wavepacket (10 points each)

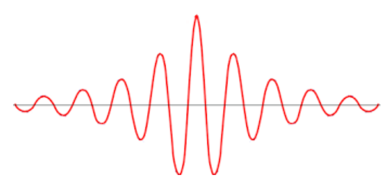
Given the **dispersion $\boldsymbol{\omega}(\boldsymbol{k})$** of a medium in which wave propagates we will clarify the difference between phase velocity $v_p = \omega/k$ and group velocity $v_g = d\omega/dk$ with examples.

a) Beats of two sinusoidal waves



Consider the superposition of two waves $\psi = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$, where k_1, k_2 are the wavenumbers and ω_1, ω_2 are the frequencies.
Show that the wavefunction can also be described by an infinite traveling wave that propagates at the "phase velocity" $v_p = (\omega_1 + \omega_2)/(k_1 + k_2)$ with an envelope function (dashed line in the figure) that propagates at the "*group velocity*" $v_g = (\omega_1 - \omega_2)/(k_1 - k_2)$.
Use this example to argue that for waves with similar wave numbers $k_1 \approx k_2$ and amplitudes, the phase velocity is $v_p = \langle\omega\rangle/\langle k\rangle$ and the group velocity is $v_g = d\omega(\langle k\rangle)/dk$, where $\langle x\rangle$ is the mean value of x.

b) Sinusoidal waves with a generic envelop function



We model a pulse, see right figure, by the product of an infinite traveling wave $e^{i(k_0 x - \omega_0 t)}$, where $\omega_0 = \omega(k_0)$ and a smooth envelop function $A(x)$.
Given the initial condition

$$\psi(x, 0) = A(x) e^{ik_0 x}$$

prove that to leading order, the wavefunction is

$$\psi(x, t) = A\big(x - v_g t\big) e^{ik_0 (x - v_p t)},$$

where phase velocity is $v_p = \frac{\omega_0}{k_0}$ and $v_g = \frac{d\omega(k_0)}{dk}$.
(Hint: You may use the Taylor expansion $\omega(k) \approx \omega(k_0) + (k - k_0)\omega'(k_0)$, write the generic solution of the wave equation in the Fourier space as $\psi(x,t) = \int f(k) e^{i[kx - \omega(k)t]} dk$, where you can identify $f(k)$ from the initial condition.)

c) A pianist is playing in a concert hall. Poor you are sitting $L = 100$ m from the stage. The air is weakly dispersive $\omega(k) \approx v_p k + \epsilon k^2$ with $\epsilon = 8 \times 10^{-4}\ m^2/s$. When the pianist plays the lowest key A_0 at $\omega_L = 2\pi \times 27.5$Hz and the highest key C_8 at $\omega_R = 2\pi \times 4186$ Hz simultaneously which note will reach your ear first and how long is the delay between the 2 notes?
(Hint: you may use the nominal sound speed at 1 atm as 343 m/s.)

2. Doppler effect and echo (10 points each)

Conventional Doppler radar guns use ultrasound to measure the velocity of a car by reflection. (Modern ones use microwaves). Assume the gun emits sound wave at frequency ω, the phase velocity is v_p and the police detects the frequency of the reflected waves as ω_r. From the frequency of the reflected waves, the police determine the speed of the car.

a) A car is moving straight away from the police at the speed $v < v_p$, show that the frequency of the reflected waves has a frequency of

$$\omega_r = \omega_0 \frac{v_p - v}{v_p + v}$$

b) The above formula is inaccurate in the presence of strong wind. Assume the wind is in the same direction as the car and the wind speed is w, show that the frequency of the reflected wave is

$$\omega_r = \omega_0 \frac{v_p - w}{v_p + w} \frac{v_p + w - v}{v_p - w + v}$$

Hint: Sound propagates in air. So when the air moves at velocity w, sound propagates faster in the same direction at velocity $v_p + w$, and slower in the opposite direction at $v_p - w$. Determine the time duration when two consecutive wavefronts are reflected by the car, and the spatial separation between them. The frequency of the echo here is determined by the duration the police receive two consecutive reflected wavefronts from the car.

c) How fast do you have to drive (a racket maybe) relative to a traffic light so the red light at wavelength $\lambda_0 = 650$ nm appears to be green at $\lambda_r = 522$ nm?

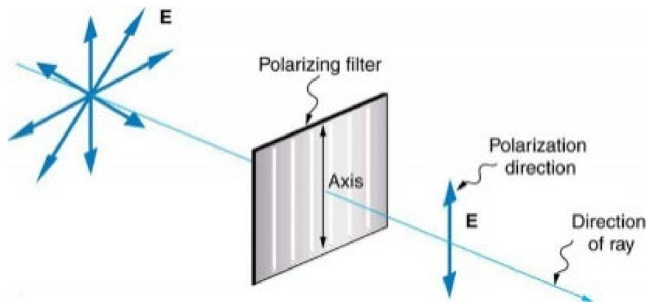
(Hint: Relativistic time dilation should be included such that the Doppler shift does not depend on the choice of the reference frame and is given by $\omega_r = \omega_0 \sqrt{\frac{c+v}{c-v}}$, where v is the relative velocity.)

d) A GPS satellite orbits Earth at an altitude of approximately 20,200 km and moves at a speed of about 3.87 km/s relative to the Earth's surface. The satellite continuously emits a signal at a carrier frequency of $f_0 = 1.57542$ GHz. (i.e., the L1 frequency used in civilian GPS).

Suppose a receiver on Earth is directly beneath the satellite's orbital path, how well should the receiver perform the frequency measurement to determine its own velocity at the sensitivity level of 1 m/s?

(Hint: For small velocity $v \ll c$, Doppler shifts can be well approximated by $\frac{\Delta f}{f} \approx \frac{v}{c}$.)

3. Polarizers (10 points each)
Polaroid polarizing filters are made of nitrocellulose polynemer film, where the polymers form needle-like crystals that only absorb light with polarization (direction of electric field) along the direction of the crystals. The axis of the polarizer is typically defined as the direction that the light can transmit. See figure, where the polarizer axis is in the vertical (x-)direction, and the light propagates in z.



a) Assume the incident beam electric field is given by $E_i = (E_x, E_y)e^{ik(z-ct)}$, and the polarizer is aligned in the x direction as shown in the figure. The transmission rejects E field in the y-direction and thus the transmitted beam is $E_t = E_x(1,0)e^{ik(z-ct)}$. If we place a second polarizer rotated by 90 degrees relative to the first one, show or argue that the transmission is zero.

b) Now we add a third polarizers between the two polarizers at an angle θ relative to the first one. Show that light can transmit again with electric field $E_i = E_x(0, \sin\theta\cos\theta)$.

c) Show that when an incident beam with electric field $E_{in} = (E_x^{in}, E_y^{in})$ passes through a polarizer rotated by $+\theta$ relative to the x-axis, the outgoing field is given by

$$\begin{pmatrix} E_x^{out} \\ E_y^{out} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{pmatrix} E_x^{in} \\ E_y^{in} \end{pmatrix}.$$

(Hint: You may use superposition principle to determine the elements of the polarizer matrix.)