Physics 143a: Honors Waves, Optics, and Heat

Spring Quarter 2025
Problem Set #9
No due date

1. **Properties of ideal gas** (10 points each)
Three important laws we learned about an ideal gas:
   - **Ideal gas law**: $PV = NkT$
   - **Internal energy of the gas**: $U = \gamma NkT$, where $\gamma = 3/2$ for an atomic gas and $\approx 5/2$ for a molecular gas.
   - **Energy conservation**: $\Delta U = \Delta Q - \Delta \mathrm{W}$, where ΔQ is the heat the system absorbs and $\Delta W = P\Delta V$ is the work done by the system.

   Show that we can derive several other interesting properties of an ideal gas:

   a) Isochoric specific heat $c_V = \frac{1}{M}\partial_T Q|_V$ is the amount of heat required to increase the temperature of the system of one unit mass by 1 degree when the volume is held constant. Determine c_V and show that we can write the internal energy of the gas as

   $$U = c_V NmT,$$

   where m is the mass of a single air molecule.

   Hint: $\partial_T Q|_V$ means $\Delta Q/\Delta T$ evaluated when the volume V is unchanged $\Delta V = 0$.

   b) Compressibility is the one we discussed when we derive the sound speed in Lectures 11 and 12. In the derivation, we introduced the compressibility as the fractional reduction of the volume per unit pressure added to the system:

   $$\beta = -\frac{1}{V}\frac{\partial V}{\partial P}$$

   There are two forms of compressibility we determine: isothermal compressibility $\beta_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$ and isentropic compressibility $\beta_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S$. Show that they are $\beta_T = \frac{1}{P}$ and $\beta_S = \frac{\gamma}{(\gamma+1)P}$ for an ideal gas.

   Here isothermal means the temperature is kept constant, while isentropic means the process is adiabatic and reversible.

   c) Thermal expansion determines the fractional change of the system volume when the temperature increases by 1 degree. There are again two possible processes: isobaric thermal expansion coefficient $\alpha_V = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$ and isochoric thermal expansion coefficient $\alpha_p = \frac{1}{P}\left(\frac{\partial P}{\partial T}\right)_V$. Show that both are the same $\alpha_V = \alpha_P = \frac{1}{T}$.

2. Maxwell-Boltzmann distribution (10 points each)
A great insight from Maxwell is that the velocity distribution of molecules in an ideal gas with mass m and velocity $\vec{v} = (v_x, v_y, v_z)$ is given by $P(\vec{v}) = p(v_x)p(v_y)p(v_z)$, where $p(v_i) \propto e^{-mv_i^2/2kT}$ is the Maxwell-Boltsmann distribution.
   a) Given the probability conservation $\int_{-\infty}^{\infty} p(x)dx = 1$ and $\iiint P(\vec{v})d^3v = 1$ for any stochastic variable x, determine the explicit form of $p(v_x)$ and $P(\vec{v})$.
   b) Show that the probability distribution of $P(v)$, where $v = |\vec{v}| \geq 0$ is the absolute value of the molecular velocity, is given by

$$\mathrm{p}(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-mv^2/2kT},$$

which is the most common form of Maxwell-Boltzmann distribution.
Hint: A coordinate transform from Cartesian coordinate (x, y, z) to spherical coordinate should satisfy probability conservation. Assume the probability to find a particle in space within a small volume element dv is $dP = P(\vec{r})dv$, the same probability should be found in a new coordinate system, namely,

$$dP = P(\vec{r})dxdydz = P(\vec{r})r^2 \sin\theta \, drd\theta d\phi \equiv P(r, \theta, \phi)drd\theta d\phi.$$

   a) Determine the root-mean-square velocity $v_{rms} = \sqrt{< v^2 >}$ of an ideal gas and show that $v_{rms} = \sqrt{\frac{3kT}{m}}$ is exactly consistent with the equipartition theorem. Show that the rms velocity is higher than the isothermal or isentropic sound velocity $v_p = \frac{1}{\sqrt{n\beta}}$, using the result from 1 (b). Is it surprising that air molecules are moving at supersonic speed?

3. **Entropy**

Consider an ideal gas with energy U can exchange energy with the environment in two ways $\Delta U = \Delta Q - P\Delta V$, heat exchange $\Delta Q = T\Delta S$ and mechanical work done to the environment $-P\Delta V$.

b) An adiabatic process is the process that forbids heat exchange $\Delta Q = 0$. Assuming the system is described by the ideal gas law $PV = NkT$ and $U = \gamma NkT$, show that an adiabatic process is described by $PV^{(\gamma+1)/\gamma} = C$ and argue that the constant C is related to the entropy S.

c) To understand the relationship between adiabatic process and entropy, we first try to determine entropy of an ideal gas. Use $dU = TdS - PdV$ and show that the entropy difference between the system in state 1 with pressure P_1, volume V_1 and in state 2 with pressure P_2, volume V_2 is given by

$$S_2 - S_1 = c_v Nm \ln \frac{P_2 V_2^{(\gamma+1)/\gamma}}{P_1 V_1^{(\gamma+1)/\gamma}}.$$

Hint: Recast the equation as a differential equation $dS = \frac{dU}{T} - \frac{PdV}{T}$. Then use the ideal gas law to replace U in terms of P and V, and then integrate the differential equation.

d) Show that this result is compatible with the Sackur-Tetrode equation for γ=3/2.

$$S = Nk \ln \frac{V}{N} \left(\frac{4\pi m}{3h^2} \frac{U}{N} \right)^{3/2}$$

Hint: This derivation yields Planck constant h. Dr. Sackur could have determined the value of *h* few decades before Max Planck did.