2025 P143A Midterm 4/21/2025

Solution

1. A. Yes b^2-4ac is more negative

B. No. 2 roots are both real or complex.

C. False Driven oscillator is always oscillating.

D. independent V'' remains a constant.

E. $\omega < \omega_0$ When damping dominates larger displacement at lower f

F. $\omega_0/2$ $M \to 2M$. $K \to K/2$ (double displacement) $\to \sqrt{K/m} \to \frac{1}{2}\sqrt{K/m}$

G much longer $b^2 - 4m\omega_0^2$ will become positive when ω_0 is low enough

H. CH_4 more particles $\to$ more modes

2. (a)

$$\begin{aligned} m_1 x_1'' &= -k_1(x_1 - x_2) \\ m_2 x_2'' &= -k_1(x_2 - x_1) - k_2(x_2 - x_3) \\ m_3 x_3'' &= -k_2(x_3 - x_2) \end{aligned} \Rightarrow \begin{aligned} x_1'' &= -\frac{k_1}{m_1}x_1 + \frac{k_1}{m_1}x_2 \\ x_2'' &= \frac{k_1}{m_2}x_1 - \left(\frac{k_1}{m_2} + \frac{k_2}{m_2}\right)x_2 + \frac{k_2}{m_2}x_3 \\ x_3'' &= \frac{k_2}{m_3}x_2 - \frac{k_2}{m_3}x_3 \end{aligned}$$

$$\Rightarrow \hat{M} = \begin{pmatrix} -k_1/m_1 & k_1/m_1 & 0 \\ k_1/m_2 & -(k_1+k_2)/m_2 & k_2/m_2 \\ 0 & k_2/m_3 & -k_2/m_3 \end{pmatrix}$$

(b) $\Rightarrow \hat{M} = \omega_0^2 \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

Let $\vec{x} = \vec{A} e^{i\omega t}$ $\quad -\omega^2 \vec{A} = \omega_0^2 \hat{M} \vec{A}$

Let $\alpha \equiv \frac{\omega}{\omega_0} \Rightarrow M\vec{A} = -\alpha^2 \vec{A} \Rightarrow \det \begin{vmatrix} -1+\alpha^2 & 1 & 0 \\ 1 & -2+\alpha^2 & 1 \\ 0 & 1 & -1+\alpha^2 \end{vmatrix} = 0$

$\Rightarrow (\alpha^2-1)(\alpha^2-2)(\alpha^2-1) - (\alpha^2-1) - (\alpha^2-1) = 0$

$\Rightarrow (\alpha^2-1)[\alpha^4 - 3\alpha^2 + 2 - 2] = 0$

$\Rightarrow (\alpha^2-1)\alpha^2(\alpha^2-3) = 0 \Rightarrow \omega = \pm\omega_0,\ 0,\ \pm\sqrt{3}\,\omega_0 \Rightarrow$ eigen freq $= \omega_0,\ \sqrt{3}\omega_0$

(c) For $\alpha = 1$ $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$ $\quad \vec{A} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\quad$ compression

For $\alpha = \sqrt{3}$ $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$ $\quad \vec{A} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\quad$ asymmetric mode

3. A. $\phi(k) = \frac{1}{2\pi}\int \phi(x) e^{-ikx}\, dx$

$= \frac{1}{2\pi}\int \frac{A}{2}(e^{igx} + e^{-igx}) e^{-ikx}\, dx$

$= \frac{1}{2\pi}\frac{A}{2} 2\pi[\delta(k-g) + \delta(k+g)] = \frac{A}{2}\delta(k-g) + \frac{A}{2}\delta(k+g)$

$\psi(x,0) = \phi(x) = \int \frac{A}{2}(\delta(k-g) + \delta(k+g)) e^{ikx}\, dk = \frac{A}{2} e^{igx} + \frac{A}{2} e^{-igx}$

B. use $Ae^{i(gx+\omega t)}$ as ansatz, we get $\omega^2 = g^2 v^2 \Rightarrow \omega = \pm g v$

$\Rightarrow \psi = c_1 e^{ig(x-vt)} + c_2 e^{-ig(x-vt)} + c_3 e^{ig(x+vt)} + c_4 e^{-ig(x+vt)}$

C. $\psi(x,0) = \frac{A}{2} e^{igx} + \frac{A}{2} e^{-igx} \Rightarrow c_1 + c_3 = A/2 \quad c_2 + c_4 = A/2$

$\dot\psi(x,0) = c_1(-i\omega)e^{igx} + c_2(i\omega)e^{-igx} + c_3(i\omega)e^{igx} + c_4(-i\omega)e^{-igx} = 0$

$\Rightarrow c_1 - c_3 = 0. \quad c_2 - c_4 = 0$

$\Rightarrow c_1 = c_2 = c_3 = c_4 = A/4$

$\Rightarrow \psi(x,t) = A/2 \cos g(x-vt) + A/2 \cos g(x+vt)$

There are 2 waves with $A/2$ and velocity $\pm v$