Physics

- Newton's law: $m\vec{x}''(t) = \vec{F}(t)$
- Hooke' law: $\vec{F}(t) = -k\vec{x}(t)$
- Stokes' drag: $\vec{F}(t) = -b\vec{x}'(t)$
- Wave equation: $\partial_t^2 \psi(x,t) = v^2 \partial_x^2 \psi(x,t)$

Math

- 2nd order ordinary differential equation
  $$ax''(t) + bx'(t) + cx(t) = f(t)$$
- Characteristic equation $a\alpha^2 + b\alpha + c\alpha = 0$
- Solution of the quadratic equation $\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Euler's formula $e^{i\phi} = \cos\phi + i\sin\phi$
- Matrix determinant $\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = ad - bc$
  $$\left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right| = a(ei - fh) - b(di - fg) + c(dh - eg)$$
- Fourier series (period L)
  $$f(x) = f(x+L) = c_0 + \sum_n c_n \cos\frac{2\pi nx}{L} + \sum_n d_n \sin\frac{2\pi nx}{L}$$
  Orthogonality condition
  $$\int_0^L \cos\frac{2\pi nx}{L}\cos\frac{2\pi mx}{L}dx = \int_0^L \sin\frac{2\pi nx}{L}\sin\frac{2\pi mx}{L}dx = \frac{L}{2}\delta_{nm}$$
  $$\int_0^L \sin\frac{2\pi mx}{L}dx = \int_0^L \cos\frac{2\pi mx}{L}dx = \int_0^L \cos\frac{2\pi nx}{L}\sin\frac{2\pi mx}{L}dx = 0$$
- Fourier and inverse Fourier transform
  $$f(k) = \frac{1}{2\pi}\int f(x)e^{-ikx}dx \qquad f(x) = \int f(k)e^{ikx}dk$$
  Orthogonality conditions:
  $$\int e^{i(k-k')x}dx = 2\pi\delta(k-k')~~f(x)~~$$
- Integrals:
  $\int \delta(x)dx = 1$ $\int f(x)\delta(x-x_0)dx = f(x_0)$
  $\int \sin mx\,dx = -\frac{1}{m}\cos mx + const.$ $\int \cos mx\,dx = \frac{1}{m}\sin mx + const.$

1. **Multiple-choice questions (5 points each)**

A particle with mass m is confined in a harmonic potential $V(x) = \frac{1}{2}kx^2$ and the friction force is $-bx'(t)$, where $k > 0$ and $b > 0$. Answer the following questions. You do not need to show the calculation.

A. If we release the particle at time $t = 0$ with $x(0) = x_0$ and $x'(0) = v_0$ and see underdamped oscillations. The motion will still be underdamped if we double the mass and keep everything else the same.
Yes or No

B. By tuning the damping coefficient b, it is possible that the oscillator can display a superposition of underdamped and overdamped motion.
Yes or No

C. In the presence of strong damping $b > 2m\omega_0$, the particle displays an overdamped motion when it is driven by a weak periodic force.
True or False

D. When the gravitational force mg applies to a vertically oriented oscillator with potential energy $V(z) = \frac{1}{2}kz^2 - mgz$, the frequency of the oscillation is
higher for larger g independent of g smaller for larger g

E. When an over-damped oscillator with natural frequency $\omega_0 \equiv \sqrt{k/m}$ is driven by a force $f(t) = f \cos \omega t$, the oscillator responds with the maximum amplitude when the driving frequency is
$\omega < \omega_0$ $\omega = \omega_0$ $\omega_0 < \omega \leq 2\omega_0$ $\omega > 2\omega_0$

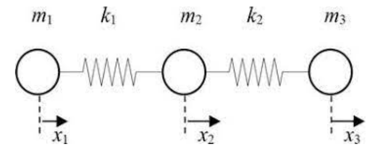
F. You pick up a massive spring vertically at one end (nothing is attached to the other end) and observe vertical oscillations of the spring with frequency ω_0 and negligible damping. Now you connect two identical springs in series and repeat the experiment, and the new oscillation frequency is
$\omega_0/2$ $\omega_0/\sqrt{2}$ ω_0 $\sqrt{2}\omega_0$ $2\omega_0$

G. A pendulum motion in air is typically underdamped. Keeping the mass and air friction coefficient constant, can one possibly make the pendulum overdamped by changing the length of the string?
No way Yes, with a much longer string Yes, with a much shorter string

H. Which molecule has more normal modes near the equilibrium?
O_2 CO_2 H_2O CH_4 NH_3

2. **Coupled oscillators (10 points each)**

Three masses are connected by 2 springs with masses and force constants given in the figure on the right. They move in one dimension.



Introduce the displace vector $\vec{x}(t) \equiv \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$,

which follows the equation of motion (we ignore damping here)

$$\vec{x}''(t) = \hat{M}\vec{x}(t)$$

A. Determine the 3x3 matrix $\hat{M}$.

B. Assume equal mass $m_1 = m_2 = m_3 \equiv m$ and equal force constant $k_1 = k_2 \equiv m\omega_0^2$. Determine all eigen-frequencies ω.

   Hint: Use ansatz $\vec{x}(t) = \vec{A}e^{i\omega t}$, where $\vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$.

C. For each normal mode with eigen frequency ω, determine $\vec{A}$ and describe the motion of each mode.

3. **Wave equation and Fourier Transform (10 points each)**
An infinite string supports a transverse wave $\psi(x,t)$, which satisfies the wave equation $\partial_t^2\psi = v^2\partial_x^2\psi$. At $t = 0$, the string forms a sinusoidal wave described by $\phi(x) = A\cos qx$, namely

$$\psi(x,0) = \phi(x)$$

and the string is stationary

$$\partial_t\psi(x,0) = 0.$$

Here we will use Fourier transform to determine subsequent motion of the string $\psi(x, t > 0)$

A. By Fourier transform $\phi(x)$, show that

$$\phi(k) = \frac{A}{2}\delta(k-q) + \frac{A}{2}\delta(k+q),$$

and this leads to

$$\psi(x,0) = \frac{A}{2}e^{iqx} + \frac{A}{2}e^{-iqx}.$$

B. The wave equation suggests the subsequent dynamics of the string is periodic in time, described by $e^{i\omega t}$. Show that $\omega = \pm qv$ and the general solution of the wavefunction is given by

$$\psi(x,t) = c_1 e^{iq(x-vt)} + c_2 e^{-iq(x-vt)} + c_3 e^{iq(x+vt)} + c_4 e^{-iq(x+vt)}$$

C. Determine all amplitudes c_1, c_2, c_3 and c_4. How many traveling waves with real amplitudes that are propagating in the string? And what are their amplitudes and velocities?