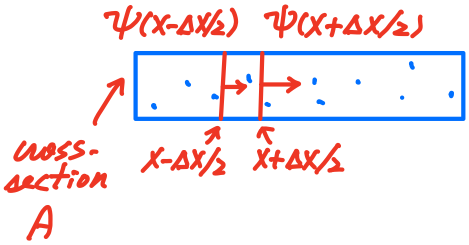
Longitudinal waves: Sound wave in gas, liquid, and solid.

step 1: What is the equilibrium state?

step 2: What's the restoring force?

step 3: Associate the force with the motion of neighbors.

Consider a uniform medium with mass density n



$$ma = n\Delta V \partial_t^2 \varphi = nA\Delta x \partial_t^2 \varphi$$

$$F = AP(x-\Delta x/2) - AP(x+\Delta x/2)$$

$$= -A\Delta x P'(x)$$

In gas and liquid $P>0$

In solid $P<0$

What is the origin of restoring force?

Let's introduce compressibility $\beta \equiv -\frac{1}{V}\partial V/\partial P$ $\left(= \frac{1}{\text{stiffness}} \right)$

$$\Delta P = -\frac{1}{\beta}\frac{\Delta V}{V} = -\frac{1}{\beta}\frac{A[\varphi(x+\Delta x/2) - \varphi(x-\Delta x/2)]}{A\Delta x}$$

pressure in equilibrium

$$\Rightarrow \boxed{P(x) - P_0 = -\frac{1}{\beta}\varphi'(x)}$$

$$n = \frac{n_0 V}{V+\Delta V} = n_0\left(1+\frac{\Delta V}{V}\right)^{-1} \approx n_0[1-\varphi'(x)]$$

Thus we have $nA\Delta x \partial_t^2 \varphi = -A\Delta x\left(-\frac{1}{\beta}\partial_x^2 \varphi\right)$

$$\Rightarrow n\partial_t^2 \varphi = \frac{1}{\beta}\partial_x^2 \varphi \quad \text{or} \quad \partial_t^2 \varphi = v^2 \partial_x^2 \varphi$$

$$\Rightarrow \boxed{v = \sqrt{1/n\beta}}$$

For an ideal gas:

$PV = NKT$. For $T = \text{const}$. $\beta = -\frac{1}{V}\partial V/\partial P = \frac{1}{V}\frac{NKT}{P^2} = \frac{1}{nKT}$

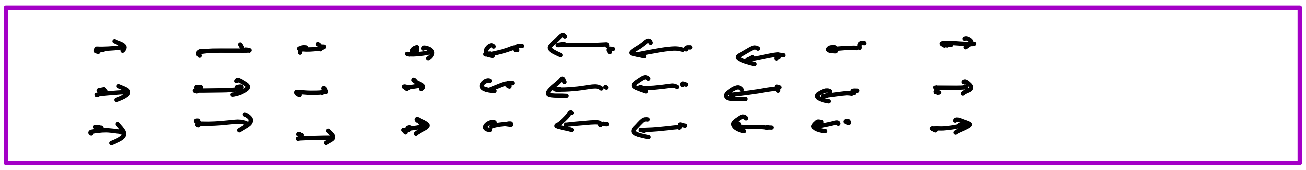
$$\Rightarrow v = \sqrt{KT} = \sqrt{\partial P/\partial n}$$

Faster sound for lower density and lower compressibility

Let's compare

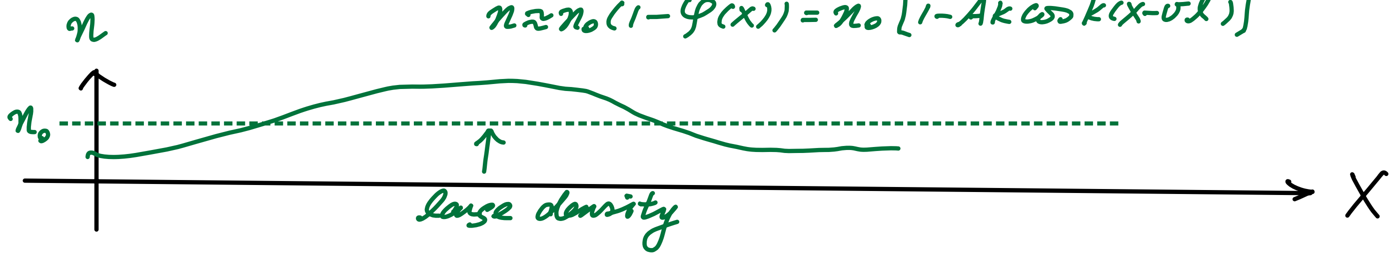
	air	water	ice
	340 m/s	1500 m/s	4000 m/s
n	small	high	higher
β	high	low	lower

Microscopic picture

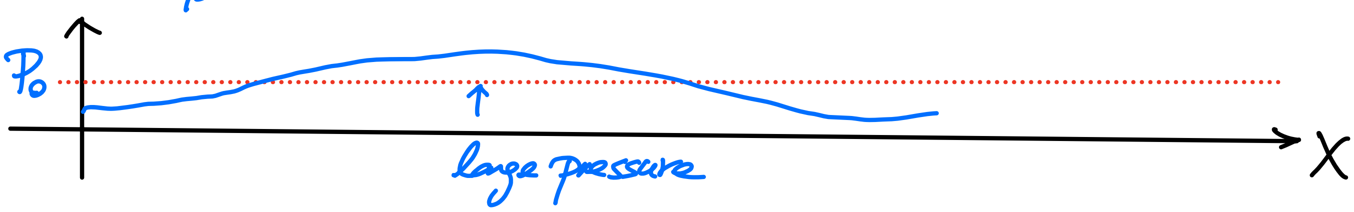


$$\varphi = A \sin k(x - vt)$$

$$n \approx n_0(1 - \varphi'(x)) = n_0\left[1 - Ak\cos k(x - vt)\right]$$



$$P = P_0 - \frac{1}{\beta} k \cos k(x - vt)$$



Electromagnetic waves

Maxwell Eqns in medium

$$\nabla \cdot E = \rho/\epsilon = 0$$ Gauss' law

$$\nabla \cdot B = 0$$ no mag. monopole

$$\nabla \times E = -\partial_t B$$ Faraday's law

$$\nabla \times B = \mu \epsilon \partial_t E + \mu_0 J = \mu \epsilon \partial_t E$$ Amp-Max. law

$$\left.\begin{aligned}\nabla \times (\nabla \times E) &= \nabla(\nabla \cdot E) - (\nabla \cdot \nabla) E = -\nabla^2 E \\ &= -\partial_t \nabla \times B = -\mu \epsilon \, \partial_t^2 E\end{aligned}\right\} \quad \partial_t^2 E = \frac{1}{\mu \epsilon} \nabla^2 E \qquad v = \frac{1}{\sqrt{\mu \epsilon}}$$

Similarly $\partial_t^2 B = \frac{1}{\mu \epsilon} \nabla^2 B$

In vacuum $\partial_t^2 \begin{Bmatrix} E \\ B \end{Bmatrix} = \frac{1}{\mu_0 \epsilon_0} \begin{Bmatrix} E \\ B \end{Bmatrix}$ $\quad v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$ $\Rightarrow$ index of refraction $n = \frac{c}{v} = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$

We may think the solution is $\vec{E}(\vec{x}, t) = \Sigma_k \vec{A}(\vec{k} \cdot \vec{x} - \omega_k t)$ $\quad \omega_k = v |\vec{k}|$

But the other Maxwell eqns provide further restriction

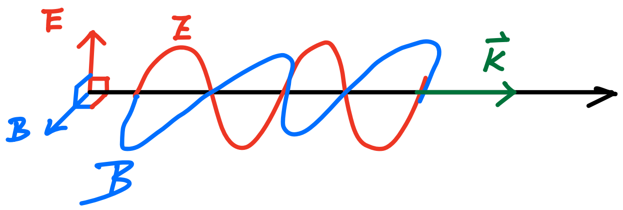
$$\vec{E}(x, t) = \vec{E} e^{i k (x - c t)} = E_j \hat{e}_j e^{i k_l x_l} e^{-i \omega t}$$

Gauss' law

$$\nabla \cdot E = \partial_j E_j e^{i k_l x_l} e^{-i \omega t} = \vec{k} \cdot \vec{E} = 0.$$

$$\nabla \cdot B = \partial_j B_j e^{i k_l x_l} e^{-i \omega t} = \vec{k} \cdot \vec{B} = 0$$

Faraday's law $\nabla \times E = -\partial_t B \Rightarrow i \vec{k} \times \vec{E} = -(-i\omega) \vec{B} \Rightarrow \hat{k} \times \vec{E} = v \vec{B}$



Forward $E^+ = (E, 0, 0)\, e^{i k (z - c t)}$

$B^+ = (0, E/c, 0)\, e^{i k (z - c t)}$

backward $E^- = (E, 0, 0)\, e^{i k (-z - c t)}$

$B^- = (0, -E/c, 0)\, e^{i k (-z - c t)}$

Standing waves: $E = E^+ + E^- = (E, 0, 0)\,[\cos k(z - ct) + \cos k(-z - ct)]$

$= (2E, 0, 0) \cos kz \cos \omega t$

$B = B^+ + B^- = (0, 2E/c, 0)\,[\cos k(z - ct) - \cos k(-z - ct)]$

$= (0, 2E/c, 0) \sin kz \sin \omega t$