Physics 143A: Honors Waves, Optics, and Heat

Spring Quarter 2025

Problem Set #1

Due: 11:59 pm, Monday, March 31. Please submit to Canvas.

1. **Ordinary Differential Equation** (5 points each)
   Solve the following problems.
   (a) $x'' + 6x' + 8x = 0$ with initial conditions $x(0) = 2$ and $v(0) = -3$.
   (b) $x'' + 3x' + 2x = e^t$ with initial conditions $x(0) = 0$ and $v(0) = 1$.
   (c) $x' + 5x = \cos t$ with initial condition $x(0) = 0$.
   (d) A bullet is fired into a swimming pool with an initial velocity $v_0 = 480$ m/s. As it moves through the water, it experiences a resistive force proportional to its velocity,
   $$F_{drag} = -b\,v,$$
   where $b=50$ kg/s is the drag coefficient. The bullet has a mass of m=15 g. Write down the equation of motion of the bullet in the form of
   $$x'' + Ax' + Bx = 0.$$
   (e) Continue (d) and determine the total distance the bullet travels before coming to rest.
   (f) (Hard) Now we fire the bullet upward with velocity v_0 and air drag is F=-b v, gravitational acceleration is g, calculate how high it can reach.

2. **Complex number** (5 points each)
   A complex number z can be expressed in the Cartesian form $z = x + iy$ and the polar form $z = re^{i\theta}$, where $i^2 = -1$ and $x, y, r, \theta \in \mathbb{R}$ are the real part, imaginary part, modulus and argument of the number, respectively. The two forms are linked by Euler's formula $x + iy = r(\cos\theta + i\sin\theta)$. Rewrite or evaluate the following complex numbers
   (a) Rewrite $z_1 = \frac{i-4}{2i-3}$ in the Cartesian form.
   (b) Rewrite $z_2 = (1+i)^\alpha$ in polar form (α is a real number).
   (c) Rewrite $z_3 = \frac{e^{i\omega t}}{\omega^2 + i\gamma\omega}$ in both the Cartesian and polar forms.
   (d) Find all distinct solutions of $z^3 + 1 = 0$.
   (e) Find all distinct solutions of $z + \frac{1}{z} = 2i$.
   (f) Find the principal value of i^i and $\ln i$.

3. **Expansion and approximation** (10 points each)
   (a) Taylor expand $f(x) = \frac{x}{1+x^2}$ to 2nd order near $x = 0$, as well as its local minimum.
   (b) Determine the asymptotic form of $f(x,y) = x^{3/2}(x+4y)^{1/2} - (x+y)^2$ when $x \gg y$. (Hint: You may rewrite the function in terms of $\epsilon \equiv y/x$ and x, and keep ϵ to leading order. At the end write your answer in terms of x and y.)
   (c) Approximate $f(x) = \frac{x-x_0}{\sqrt{(x^2-x_0^2)^2+4\gamma^2x^2}}$ when x is very close to x_0.
   (Hint: You may introduce $\epsilon \equiv x - x_0 \ll x, x_0$ and keep ϵ to leading order. At the end write your answer in terms of x and x_0.)
   (d) A particle with mass $m = 1$ is trapped in the local minimum of the potential $V(x) = \frac{x-1}{x^2+3}$. Determine the position and potential energy of the local minimum. Taylor expand the potential at the minima and determine the vibrational frequency of the particle.