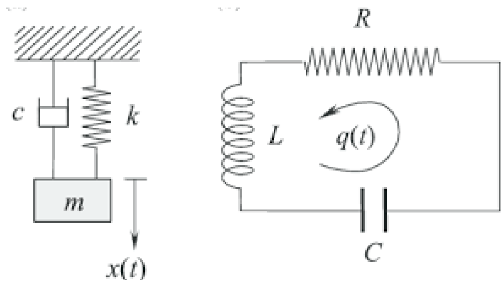
Physics 143A: Honors Waves, Optics, and Heat

Spring Quarter 2025
Problem Set #2
Due: 11:59 pm, Tuesday, April 8. Please submit to Canvas.

1. **Simple harmonic oscillators in mechanics and electronics (10 points each)**

Common oscillators in mechanics and electrodynamics are spring-mass system and RLC circuit, see figure on the right.



(Left) Consider a mass m is connected to a spring which exerts a force on the mass $f = -kx(t)$, where k is the force constant. In addition, a damping force $f = -cv(t)$ on the mass dissipates the energy and an external force $F(t)$ can be applied to the mass.

(a) Show that the equation of motion on the mass is

$$mx''(t) + cx'(t) + kx(t) = F(t)$$

(b) Show that the intrinsic frequency of the oscillator is $\omega_0 = \sqrt{\frac{k}{m}}$, and for the oscillator to show an underdamped oscillation, the damping coefficient should be $c < 2\sqrt{km}$
(Assume *F(t)=0*).

(Right) Voltages across a capacitor, resistor and inductor are $V_c = \frac{q}{c}$, $V_R = IR$, and $V_L = L\frac{dI}{dt}$, where q is the charge across the capacitor, $I = dQ/dt$ is the current flowing through the circuit, and the voltage drop over the wire loop is zero $V_c + V_R + V_L$=0.

(c) Derive the differential equation of the charge $q(t)$ and the current $I(t)$ and show that they follow the equation of a damped harmonic oscillator, and the intrinsic frequency is $\omega_0 = \frac{1}{\sqrt{LC}}$.

(d) Assume $R = 0, q(0) = q_0$ and $I(0) = 0$, determine $q(t)$ and $I(t)$ of the circuit.

2. **Quality factor of an oscillator**

Crystal oscillators and atomic blocks are excellent time keepers and they operates under the principle of a driven harmonic oscillator with a low damping $\gamma \ll \omega_0$.

A driven RLC circuit equation is powered by a time-dependent voltage $V(t) = V \sin \omega t$, the equation is given by

$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = V(t)$$

(a) Derive the particular solution starting with the ansatz of $q(t) = Qe^{i\omega t}$.

(b) The current through the circuit can be written as $I(t) = I(\omega)\sin(\omega t + \phi)$. Use any software to plot $I(\omega)$. Show that the profile $I(\omega) \approx I_{max}\sqrt{L(\omega)}$ is described by the Lorentz distribution $L(\omega) = \frac{1}{1+\frac{(\omega-\omega_0)^2}{\Gamma^2}}$, when $R/L \ll \omega_0$ and $\omega \approx \omega_0$. Determine the width of the Lorentz distribution Γ, the maximum current I_{max}, and the ratio of the resonance frequency and the width $Q = \frac{\omega_0}{\Gamma}$, also called the Quality factor.

(Hint: for a bell-shaped distribution, width is typically defined as the range where the signal is above 50% of the maximum.)

(c) Plot the phase shift $\phi(\omega)$ for $R/L = \omega_0/10$ and comment on the value of the phase shift when the driving force is far below resonance $\omega \ll \omega_0$, near the resonance $\omega = \omega_0$ and well far above the resonance $\omega \gg \omega_0$.

3. **Math** Determine the eigenvalues and eigenvectors of the following matrices (5 points each)

(a) $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix}$

(c) A particle of mass 1 is moving in on the plane with potential energy $V(x,y) = x^2 + y^2 - xy - 6x$. Determine the equilibrium position (x_0, y_0), where the potential energy is at the minimum, and the two eigen-frequencies of the particle moving near the potential minimum.

(Hint: Taylor expand the potential near the minima. You may introduce a new coordinate $u = x - x_0, v = y - y_0$ and show that the equation of motion $m\vec{r}'' = -\nabla V(\vec{r})$ is given by

$$u'' = -2u + v$$
$$v'' = -2v + u.$$

You can then derive the eigenfrequencies.)

(d) Use the result in (c) and determine the eigen-frequencies of the particles moving near the minimum of the potential $V(x,y) = e^{x^2+y^2-xy}$.