Physics 143A: Honors Waves, Optics, and Heat

Spring Quarter 2025

Problem Set #3

Due: 11:59 pm, Tuesday, April 15. Please submit to Canvas.

1. **Damping of coupled oscillators** (10 points each)

   Here we attempt to couple two overdamped oscillators and see if we can undamp the oscillators?

   We start with two identical oscillators described by $x'' + 4x' + 3x = 0$ and $y'' + 4y' + 3y = 0$. They are overdamped since $b^2 - 4ac = 4^2 - 4 \times 3$>0. Now we couple them as

$$x'' + 4x' + 3x = F_{xy}$$
$$y'' + 4y' + 3y = F_{yx},$$

   where $F_{xy} = 2(y - x)$ is the force of the y oscillator on the x oscillator. $F_{yx} = -F_{yx}$ is the reaction force.

   (A) Write the equations in the vector-matrix form as $\vec{x}''(t) + \hat{\gamma}\vec{x}'(t) + \widehat{M}\vec{x}(t) = 0$, where $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and determine the matrices $\hat{\gamma}\ and\ \widehat{M}$.

   (B) Use the ansatz $\vec{x} = \vec{A}e^{i\omega t}$, and determine the four eigen-frequencies $\omega_1, \omega_2, \omega_3, \omega_4$ and the associated amplitudes of the normal modes $\vec{A}_1, \vec{A}_2, \vec{A}_3, \vec{A}_4$. Please clearly pair the eigen-frequencies and the amplitudes. Indicate whether the mode is underdamped, critically damped or overdamped.

   (Hint: You may use a math program to calculate the eigenvalues and vectors.)

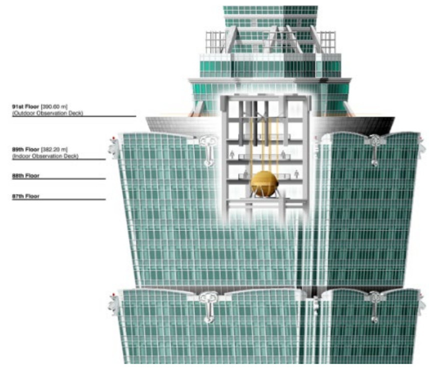
   (C) The general solution of the coupled oscillators is the linear superposition of the eigenmodes

$$\vec{x}(t) = a\,\vec{A}_1 e^{\alpha_1 t} + \ b\,\vec{A}_2 e^{\alpha_2 t} + c\,\vec{A}_3 e^{\alpha_3 t} + d\,\vec{A}_4 e^{\alpha_4 t},$$

   where a, b, c, d are arbitrary constants. Given the initial conditions $x(0) = 1, x'(0) = 0, y(0) = -1, y'(0) = 0$, determine the solution $x(t)$ and $y(t)$.

   (Hint: You may use a math program to determine the constants.)

2. **Tuned mass damper (TMD) (10 points each)**



A tuned mass damper (TMD) is an ingenious design to reduce the amplitude of vibrations in buildings or bridges due to earthquakes and storms. It typically consists of a mass, a spring, and an adjustable damper, and is attached to a vibrating structure. The figure shows the 660-ton TMD on the 91 floor of the Taipei 101 building.

Let's figure out how it works. Vibration of the building with mass M due to external force $F(t)$ can be modeled by a driven oscillator $x'' + \gamma x' + \omega_0^2 x = F(t)/M$. Given a resonant driving, the inertial force Mx'' cancels the restoring force $-\omega_0^2 x$ and the vibrational amplitude $A = \frac{F}{M\gamma\omega_0}$ diverges for systems with low damping $\gamma \to 0$, and the building is doomed.

Now we attach the TMD $y(t)$ to the building to mitigate the instability.

(A) Draw a spring-mass model for the building and TMD as 2 coupled oscillators, and show that the system can be described by the equations

$$x'' + \gamma x' + \omega_0^2 x = \frac{F(t)}{M} + \frac{f_{TMD}(t)}{M}$$

$$y'' = \frac{-f_{TMD}(t)}{m},$$

where $f_{TMD}(t) = mc(y' - x') + m\omega_1^2(y - x)$ is the force TMD applies to the building, and $-f_{TMD}(t)$ is the reaction force, M=700,000 ton and $m = 660$ ton is the TMD mass. Here $y - x$, c, and ω_1 are the relative displacement, damping and intrinsic frequency of the TMD.

(B) Write the equations in the vector-matrix form as $\vec{x}''(t) + \hat{\gamma}\vec{x}'(t) + \hat{M}\vec{x}(t) = \vec{F}(t)$, where $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and determine the matrices $\hat{\gamma}, \hat{M}$ *and* $\vec{F}(t)$.

(C) Assume the external force is exactly on resonance of the building $F(t) = Fe^{i\omega_0 t}$. Show that the steady state amplitude of the building is

$$A_x = \left[i\gamma - \epsilon\omega_0 \frac{\omega_1^2 + ic\omega_0}{\omega_1^2 - \omega_0^2 + ic\omega_0}\right]^{-1} \frac{F}{M\omega_0} \equiv \frac{A}{\xi}$$

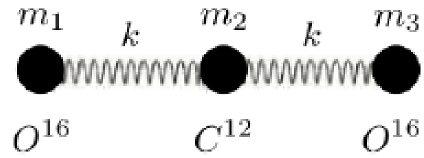
(Hint: Use the ansatz $\vec{x} = \vec{A}e^{i\omega_0 t}$, $\vec{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix}$, and determine the amplitude of A_x.)

(D) Show that when the intrinsic frequency of the TMD is tuned to the building frequency $\omega_1 \to = \omega_0$, the building vibration amplitude $|A|$ can be reduced to $|A_x|$. Keep $\epsilon \equiv \frac{m}{M} \ll 1$ to leading order and show that the suppression factor is

$$|\xi| \equiv \frac{|A|}{|A_x|} = 1 + \epsilon Q Q_{TMD} + O(\epsilon^2),$$

where $Q = \frac{\omega_0}{\gamma}$ is the quality factor of the building and $Q_{TMD} = \frac{\omega_0}{c}$ is the quality factor of the TMD. Thus suppress the vibration by a factor of > 2, we need $QQ_{TMD} > \frac{M}{m}$.

3. **Normal modes of a CO2 molecule (10 points each)**
A CO_2 molecule can be modeled as a central carbon atom with mass m_2 connected by 2 identical springs with spring constant k to two oxygen atoms with mass $m_1 = m_3 \equiv m$.



(A) First we consider linear motion of the atoms along the molecular axis. Write down the differential equations that describe their motion x_1, x_2 and x_3 and determine the eigenfrequencies in terms of m, m_2 and k.
(B) Describe the eigenmodes. In principle there should be 3 eigenmodes for 3 oscillators. Explain why you only get two vibrational modes, and where is the third one?
(C) **1.1%** of naturally occurring carbon atoms on Earth are C^{13}. How much in fraction are the two eigenfrequencies of $C^{13}O_2$ shifted relative to the regular $C^{12}O_2$?

Hint: Atomic Masses (in unified atomic mass units, u):

Isotope	**Symbol**	**Atomic Mass (u)**
Carbon-12	**^{12}C**	**12.000000 u (defined exactly)**
Carbon-13	**^{13}C**	**13.003355 u**
Oxygen-16	**^{16}O**	**15.994915 u**