Physics 143A: Honors Waves, Optics, and Thermo

Spring Quarter 2024
Problem Set #4
Due: 11:59 pm, Tuesday, April 22. Please submit to Canvas.

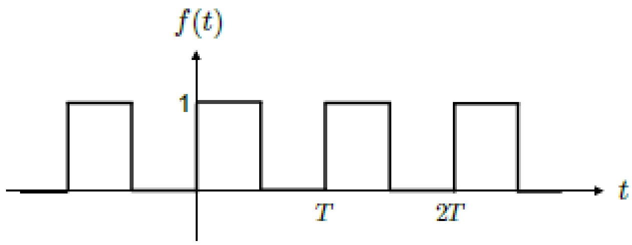
1. **(Math) Fourier series expansion and Fourier transform exercise (5 points each)**
You may Fourier expand a periodic function or Fourier transform a generic function as

Fourier series expansion: $y(x) = y(x+L) = a_0 + \sum_{n=1} a_n \cos\frac{2\pi n x}{L} + \sum_{n=1} b_n \sin\frac{2\pi n x}{L}$
Fourier transform: $y(x) = \int y(k) e^{ikx} dk$

Determine the Fourier series of the following functions
(a) Fourier expand $y(x) = |\sin x|$
Hint: first determine the period of the function $L = \pi$ and then determine the basis you need to expand it.
(b) Fourier expand an infinite series of periodic impulses
$y(x) = \sum_{n=0,\pm1,\pm2\ldots} \delta(x-n)$, where $\delta(x)$ is Dirac's Delta function.
(c) Fourier transform a single impulse $y(x) = \delta(x)$
(d) Show that given $f(x) = \int f(k) e^{ikx} dk$, we have the following normalization condition

$$\int f^*(x) f(x) dx = 2\pi \int f^*(k) f(k) dk$$

Hint: Use the formula $\int e^{ikx} dx = 2\pi\delta(k)$
(e) Consider a periodic function $f(t) = f(t+T)$, where T is the period. See below



We will expand it in 3 ways and compare the results.
Expand $f(t)$ with the trigonometric functions of the same period T

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos\frac{2\pi n t}{T} + b_n \sin\frac{2\pi n t}{T}).$$

Determine a_0, a_n and b_n. These coefficients should be real.
(f) Continue (e), expand $f(t)$ with the exponential series of the same period

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi n t}{T}}.$$

Determine c_n, which can be complex.
(g) Continue (f), Fourier transform $f(t)$ over the entire range

$$f(t) = \int f(\omega) e^{i\omega t} d\omega.$$

Determine $f(\omega)$, which can be complex, too.
(h) All these expansions from (e-g) should be equivalent. Show that c_n and $f(\omega)$ can be expressed in terms of a_0, a_n and b_n.

2. **General solution of a driven harmonic oscillator (15 points each)**
The Fourier transform allows us to solve the ODEs with arbitrary driving force.
(a) A driven oscillator is described by equation of motion:

$$x''(t) + \gamma x'(t) + \omega_0^2 x(t) = f(\text{t})$$

Show that the particular solution is

$$x(t) = \frac{1}{2\pi}\int\int \frac{f(\tau)e^{i\omega(t-\tau)}}{\omega_0^2-\omega^2+i\gamma\omega}d\omega d\tau.$$

(b) As an example, we consider the oscillator driven by the step-function force $f(t)$, described in Question 1 (f). Determine the solution of the oscillator motion.
Hint: $x(t) = \frac{1}{2\omega_0^2} + \frac{1}{\pi}\sum_{n=1,3,5\dots}\frac{1}{n}Im[\frac{e^{i\omega_n t}}{\omega_0^2-\omega_n^2+i\gamma\omega_n}]$, where $\omega_n = \frac{2\pi n}{T}$.

3. **Damped wave equation and guitar string (10 points each)**
A realistic guitar string moves according to the following wave equation with linear density ρ, damping coefficient b and elastic coefficient κ (kappa):

$$\rho\partial_t^2\psi(x,t) + b\partial_t\psi(x,t) = \kappa\partial_x^2\psi(x,t).$$

Two ends of the guitar string are fixed $\psi(0,t) = \psi(L,t) = 0$.

(a) Determine the dispersion $\omega(k)$ of the eigenmodes, where ω is the angular frequency of the wave and k is the wavenumber.

(Hint: Dispersion is the relation between angular frequency ω and wave number k. You may use the ansatz $\psi = Ae^{ikx}e^{i\tilde{\omega}t}$ and show that the eigenfrequency is $\omega = Re[\tilde{\omega}] = \sqrt{\frac{\kappa}{\rho}k^2 - \frac{b^2}{4\rho^2}}$. Next show that to satisfy the boundary condition the wavefunction should have the form $\psi = A\sin kx\, e^{i\omega t}$ with the wavenumber $k = k_n = \frac{(n+1)\pi}{\text{L}}$ which takes on discrete values *n*=0,1,2,3….)

(b) Assume the initial position and velocity of the string are given by

$$\psi(x,0) = f(x), \qquad \partial_t\psi(x,0) = g(x)$$

Consider a frictionless string $b = 0$ for simplicity, determine the wavefunction $\psi(x,t)$ for $t > 0$ and verify that the solution is real when both $f(x)$ and $g(x)$ are real.

(c) A string instrument can produce a rich spectrum with fundamental ω_0 and unique overtones $\omega_n, n = 1, 2$ …. Without dissipation $b = 0$ the pitch of the overtone $\omega_n = (n+1)\omega_0$ is an exact multiple of the fundamental.
In the presence of small damping $0 < b \ll \sqrt{\kappa\rho/L^2}$, however, the overtones are detuned from the exact multiple of the fundamental $\omega_n^* = (n+1)\omega_0^* + \delta_n$. Show that the detuning

$$\delta_n \approx (n + 1 - \frac{1}{n+1})\frac{b^2}{8\rho^2\omega_0}$$

is severe for lower frequencies.
(Hint: in the presence of damping, the fundamental frequency is also shifted.)