Physics 143A: Honors Waves, Optics, and Thermo

Spring Quarter 2025
Problem Set #5
Due: 11:59 pm, Tuesday, April 29. Please submit to Canvas.

1. **Dirac's Delta function** $\delta(x)$ (5 points each)
   We may formally introduce Dirac's Delta function in the following
   - $f(x)$ is any smooth function that has an integrated area of $\int f(x)dx = 1$.
   - Dirac's delta function is defined as $\delta(x) = \lim_{\Delta\to 0} \frac{1}{\Delta} f\left(\frac{x}{\Delta}\right)$

   (a) A common choice of f by physicists is based on the Gaussian function $f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$. Apply the above definition and prove that $\delta(x)$ satisfies the following properties
   1. $\delta(x \neq 0) = 0$
   2. $\delta(x = 0)$ diverges
   3. $\int \delta(x)dx = 1$
   4. $f(x) = \int f(u)\delta(x-u)du.$

   (b) Calculate the following
   1. $\int g(x)\delta(ax+b)dx$
   2. $\int g(x)\frac{d\delta(x)}{dx}\,dx, \quad g(\pm\infty) = 0$

2. **Green's function (10 points each)**
   Let's consider a new approach to solve the driven harmonic oscillator with arbitrary force $f(t)$ described by $x'' + \gamma x' + \omega_0^2 x = f(t)$. We assume the oscillator is at rest in the far past $x(t=-\infty) = x'(t=-\infty) = 0$.

   First, let's consider the case where the external force contains a single instantaneous impulse at times $t = \tau$, given by the delta function $f(t) = \delta(t-\tau)$.

   (a) Show that immediately after the impulse the position and velocity of the oscillator is $x(\tau) = 0$ and $x'(\tau) = 1$.
   Show that the solution of the oscillator with $\gamma = 0$ is
   $$x_G(t-\tau) \equiv \begin{cases} 0, & t \leq \tau \\ \dfrac{1}{\omega_0}\sin\omega_0(t-\tau), & t \geq \tau \end{cases}$$
   The response of the system $x_G(t)$ due to an impulse at $t = 0$ is called the Green's function.
   Hint: integrate the equation of motion with the external force $f(t) = \delta(t-\tau)$.

   (b) **Green's function theorem** From the result of Question 1 (a) 4, we learned that a force can always be written as a summation of many impulses
   $$f(t) = \int f(\tau)\delta(t-\tau)d\tau.$$
   From the superposition principle: *if $x_1(t)$ is the solution for the external force $f_1(t)$ and $x_2(t)$ is the solution for the external force $f_2(t)$, then $x_1(t) + x\_2(t)$ is the solution for the external force $f_1(t) + f_2(t)$*, show that the general solution of the oscillator is

$$x(t) = \int_{-\infty}^{t} F(\tau) x_G(t-\tau) d\tau = \frac{1}{\omega_0} \int_{-\infty}^{t} F(\tau) \sin \omega_0 (t-\tau)\, d\tau$$

(c) Argue that you can apply the Green's function method to solve any linear differential equation $\hat{L}x(t) = f(t)$ using the Green's function method, where $\hat{L}$ is any linear differential operator satisfying $\hat{L}(x_1 + x_2) = \hat{L}x_1 + \hat{L}x_2$. Is this approach consistent with the one based on Fourier transform described in HW4 2 (a)?

3. Energy and energy flow of waves (10 points each)

Here we will investigate how waves transport energy in a medium (string, air, water…). Assume the wave (transverse or longitudinal) satisfies the following wave equation

$$\rho \partial_t^2 \psi(x,t) = T \partial_x^2 \psi(x,t),$$

where ρ is the linear density of the medium and T characterizes the restoring force of the medium. A traveling wave propagating along the $+x$ direction is given by $\psi(x,t) = A \cos k(x - vt)$ and $v = \sqrt{T/\rho}$ is the wave propagation speed.

(a) Consider a small section between $x - \frac{\Delta x}{2}$ and $x + \frac{\Delta x}{2}$, show that the energy density include

kinetic energy density: $\rho_K = \frac{1}{2}\rho(\partial_t \psi)^2$

potential energy density: $\rho_U = \frac{1}{2}T(\partial_x \psi)^2$.

(Hint: Kinetic energy is given by $\frac{1}{2}mV^2$ of the section. As for potential energy, think about how you derive potential energy $\frac{kx^2}{2}$ from Hook's law.)

(b) Given the traveling wave solution $\psi(x,t)$, calculate the total energy densities ρ_K and ρ_U. Show that the energy is propagating at the same velocity as the wave propagation.

(c) Show that the total energy is conserved as time evolves.

(d) At some point the total energy of the section becomes zero $\rho_E = \rho_K + \rho_U$=0. Where does the energy go?