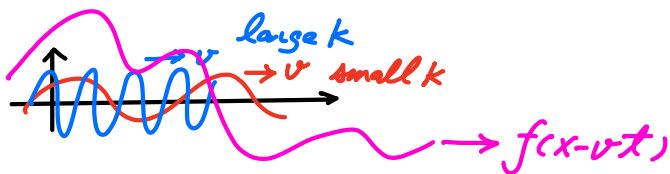


Few remarks on wave eqn and Fourier transform

$$\partial_t^2 \varphi = v^2 \partial_x^2 \varphi \quad \text{general solution is } \varphi = f(x-vt) + g(x+vt)$$

We may decompose $f(x)$ and $g(x)$ in Fourier space

$$f(x) = \int f(k) e^{ikx} dk \Rightarrow f(x-vt) = \int f(k) e^{ik(x-vt)} dk$$



$$\text{Initial condition } \varphi(x,0) = \underbrace{\int f(k) e^{ikx}}_{\text{forward moving}} + \underbrace{\int g(k) e^{ikx}}_{\text{backward moving}} dk$$

$$\text{Late time solution: } \varphi(x,t) = \underbrace{\int f(k)}_{\text{amp}} \underbrace{e^{ik(x-vt)}}_{\text{basis}} + \underbrace{\int g(k)}_{\text{amp}} \underbrace{e^{ik(x+vt)}}_{\text{basis}} dk$$

also called plane wave

What if the wave is propagating in other directions?

$$\begin{aligned} e^{ik(x'-vt)} &= e^{ik(x \cos \theta + y \sin \theta - vt)} \\ &= e^{i(K \cos \theta x + K \sin \theta y - vt)} \\ &\equiv e^{i(\vec{k} \cdot \vec{x} - \omega t)} \end{aligned}$$

Wave number $\vec{k} = (k_x, k_y)$ is a vector in the propagation direction.

$\Rightarrow e^{i(\vec{k} \cdot \vec{r} - \omega t)}$ describes wave moving in $\vec{k}$ with wave # $|\vec{k}| = 2\pi/\lambda$

$$\Rightarrow \partial_t^2 \varphi = v^2 (\partial_x^2 + \partial_y^2) \varphi$$

$$\partial_t^2 \varphi = v^2 \nabla^2 \varphi \quad \text{in 2-dimensions.}$$

Double check: $\partial_t^2 \varphi = \omega^2 \varphi$

$$\partial_x^2 \varphi + \partial_y^2 \varphi = (\cos^2 \theta + \sin^2 \theta) k^2 \varphi$$

$$\Rightarrow \omega^2 = v^2 k^2$$

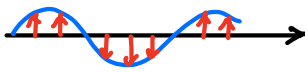
Wave phenomena: longitudinal and transverse waves.

Longitudinal

 propagation Displacement along the propagation

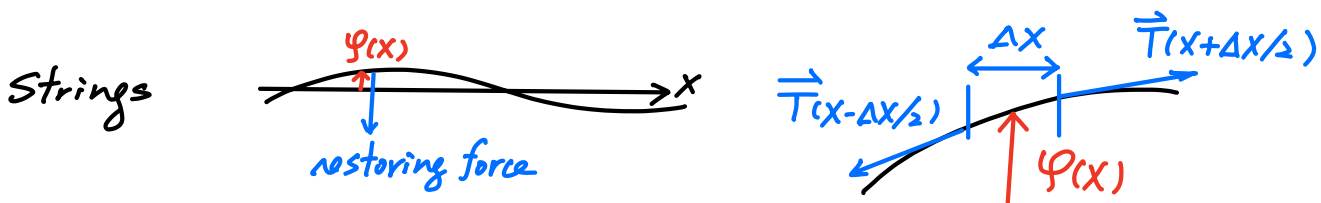
Examples: sound, spring, seismic P-wave

Transverse



Displacement transverse to the propagation

Examples: Strings, EM waves, seismic S-wave, water waves



Tension $\vec{T} = (T_x, T_y)$ is along the string $T_y/T_x = y'(x)$

$$\Rightarrow \vec{T}(x-\Delta x/2) = -(T_1, T_1 y'(x-\Delta x/2))$$

$$\vec{T}(x+\Delta x/2) = (T_2, T_2 y'(x+\Delta x/2))$$

no transverse motion: $\sum \vec{T}_x = T_2 - T_1 = 0$

yes longitudinal motion: $\sum \vec{T}_y = T y'(x+\Delta x/2) - T y'(x-\Delta x/2) = \rho \Delta x \partial_t^2 y$

$$\Rightarrow \rho \Delta x \partial_t^2 y = T (y'(x+\Delta x/2) - y'(x-\Delta x/2))$$

$$= T \Delta x \partial_x^2 y$$

$$\Rightarrow \rho \partial_t^2 y = T \partial_x^2 y \quad \text{Compared with } \partial_t^2 y = v^2 \partial_x^2 y$$

$$\Rightarrow \text{String velocity } v = \frac{\omega}{k} = \sqrt{T/\rho}$$

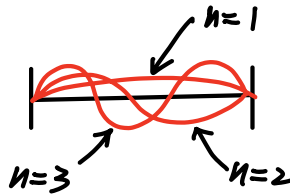
If length is fixed $k = \text{const.}$

higher tension higher pitch $\omega \propto \sqrt{T}$

higher density lower pitch $\omega \propto \sqrt{1/\rho}$

Boundary conditions

1. Fixed $\psi(x=0) = \psi(x=L) = 0$

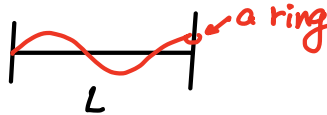


$$\psi(x) = \sum_{n=1}^{\infty} a_n \sin n \frac{\pi x}{L}$$

How about time dependence? Each wave goes like $e^{\pm i k x} e^{\pm i \omega t}$
with $v = \frac{\omega}{k}$ given by $\sqrt{T/\rho}$

$$\begin{aligned} \psi(x, t) &= \sum_{n=1}^{\infty} a_n \sin \frac{n \pi x}{L} (b_n e^{i k_n x} + c_n e^{-i k_n x}) \\ &= \sum_{n=1}^{\infty} \sin \frac{n \pi x}{L} (b'_n \cos \frac{n \pi v t}{L} + c'_n \sin \frac{n \pi v t}{L}) \end{aligned}$$

2. Open boundary condition



$$L = \frac{1}{4} \lambda, \frac{3}{4} \lambda, \frac{5}{4} \lambda \dots$$

$$= \frac{2n+1}{4} \lambda \quad n=0, 1, 2 \dots$$

another way to write the solution

$$\psi = \sum_{n=0}^{\infty} a_n \sin k_n x \cos(\omega_n t + \phi_n)$$

$$k_n = \frac{2\pi}{\lambda} = 2\pi \frac{2n+1}{4L}$$

3. Free boundary condition

$$\psi = \sum_n a_n e^{i k_n (x-vt)} + b_n e^{i k_n (x+vt)} \rightarrow \int \psi_+(k) e^{i k (x-vt)} + \psi_-(k) e^{i k (x+vt)} dk$$

forward waves backward waves.

4. Connecting two strings of diff. density ρ_1, ρ_2



Waves going to diff medium
Water, light, seismic and others.

What do we do @ the boundary?