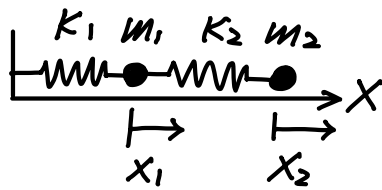


Coupled oscillators

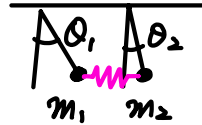
Example 1



$$m_1 x_1'' = -K_1 x_1 - K_2 (x_1 - x_2) - b_1 x_1'$$

$$m_2 x_2'' = -K_2 (x_2 - x_1) - b_2 x_2'$$

Example 2



$$\begin{matrix} A & B \\ \bigcirc & \text{---} & \bigcirc \\ f_{BA} & = & -f_{AB} \end{matrix}$$

action & reaction forces

step 1: organize the terms

$$m_1 x_1'' + b_1 x_1' + (K_1 + K_2) x_1 - K_2 x_2 = 0$$

$$m_2 x_2'' + b_2 x_2' - K_2 x_1 + K_2 x_2 = 0$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' + \begin{pmatrix} b_1/m_1 & 0 \\ 0 & b_2/m_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' + \begin{pmatrix} (K_1 + K_2)/m_1 & -K_2/m_1 \\ -K_2/m_2 & K_2/m_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\text{Define } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \vec{x}'' = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix}$$

$$\Rightarrow \vec{x}'' + \hat{\gamma} \vec{x}' + \hat{M} \vec{x} = 0$$

$$\text{Ansatz: } \vec{x} = \vec{A} e^{\alpha t}, \text{ explicitly } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{\alpha t}$$

$$\vec{x}' = A \alpha e^{\alpha t}$$

$$\vec{x}'' = A \alpha^2 e^{\alpha t}$$

$$\Rightarrow \alpha^2 e^{\alpha t} \vec{A} + \alpha \hat{\gamma} \vec{A} e^{\alpha t} + \hat{M} \vec{A} e^{\alpha t} = 0$$

$$\Rightarrow (\alpha^2 \hat{I} + \alpha \hat{\gamma} + \hat{M}) \vec{A} = 0$$

$$\hat{I}: \text{identity matrix } \hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Two options: $\vec{A} = 0 \Rightarrow \text{equilibrium}$

$$|\alpha^2 \hat{I} + \alpha \hat{\gamma} + \hat{M}| = 0$$

$$\text{Given a matrix } \hat{N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|\hat{N}| = \det \hat{N} = ad - bc.$$

$$\left| \begin{pmatrix} \alpha^2 + \alpha b_1/m_1 + (K_1 + K_2)/m_1 & -K_2/m_1 \\ -K_2/m_2 & \alpha^2 + \alpha b_2/m_2 + K_2/m_2 \end{pmatrix} \right| = 0$$

$$\Rightarrow (\alpha^2 + \frac{b_1}{m_1} \alpha + \frac{K_1 + K_2}{m_1}) (\alpha^2 + \frac{b_2}{m_2} \alpha + \frac{K_2}{m_2}) - \frac{K_1 K_2}{m_1 m_2} = 0$$

$$\Rightarrow \alpha^4 + \dots \alpha^3 + \dots \alpha^2 + \dots \alpha + \dots = 0$$

$$\Rightarrow \alpha = \alpha_1, \alpha_2, \alpha_3, \alpha_4.$$

For each α_i , we can solve for A_i such that $(\alpha_i^2 \hat{I} + \alpha_i \hat{\gamma} + \hat{M}) A_i = 0$

$$\Rightarrow \vec{X} = C_1 \vec{A}_1 e^{\alpha_1 t} + C_2 \vec{A}_2 e^{\alpha_2 t} + C_3 \vec{A}_3 e^{\alpha_3 t} + C_4 \vec{A}_4 e^{\alpha_4 t}$$

Example: $m_1 = m_2 = m$, $\gamma_1 = \gamma_2 = 0$, $K_1/m_1 = \omega$, $K_2/m_2 = \omega$

$$\Rightarrow [\alpha^2 \hat{I} + \begin{pmatrix} \omega^2 + \omega^2 & -\omega^2 \\ -\omega^2 & \omega^2 \end{pmatrix}] \vec{A} = 0 \Rightarrow \begin{pmatrix} 2\omega^2 & -\omega^2 \\ -\omega^2 & \omega^2 \end{pmatrix} \vec{A} = -\alpha^2 \vec{A}$$

eigenvalue-eigenvector problem

$$\begin{vmatrix} \alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & \alpha^2 + \omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (\alpha^2 + 2\omega^2)(\alpha^2 + \omega^2) - \omega^4 = 0$$

$$\Rightarrow \alpha^4 + 3\omega^2 \alpha^2 + \omega^4 = 0$$

$$\Rightarrow \alpha^2 = -\frac{3}{2}\omega^2 \pm \frac{1}{2}\sqrt{(3\omega^2)^2 - 4\omega^4}$$

$$= -\frac{3}{2}\omega^2 \pm \frac{\omega^2}{2}\sqrt{5}$$

$$= \frac{1}{2}(-3 \pm \sqrt{5})\omega^2 < 0 \Rightarrow \alpha = \pm \frac{i}{2}\sqrt{3+\sqrt{5}} \omega, \pm \frac{i}{2}\sqrt{3-\sqrt{5}} \omega$$

$$\text{For } \alpha_{\pm}^2 = \frac{1}{2}(-3 \pm \sqrt{5})\omega^2 \text{ we have } \begin{pmatrix} \alpha_{\pm}^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & \alpha_{\pm}^2 + \omega^2 \end{pmatrix} \vec{A}_{\pm} = 0. \Rightarrow \vec{A}_{\pm} = \begin{pmatrix} 1 \\ \frac{1}{2}(1 \pm \sqrt{5}) \end{pmatrix}$$

$$\Rightarrow \vec{X} = C_1 \vec{A}_+ e^{i\omega_+ t} + C_2 \vec{A}_+ e^{-i\omega_+ t} + C_3 \vec{A}_- e^{-i\omega_- t} + C_4 \vec{A}_- e^{i\omega_- t}$$

$$= \vec{A}_+ [C_1 \cos \omega_+ t + C_2 \sin \omega_+ t] + \vec{A}_- [C_3 \cos \omega_- t + C_4 \sin \omega_- t]$$

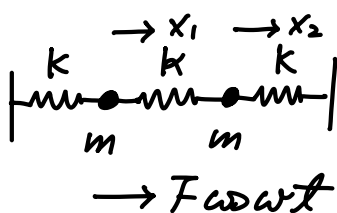
Remark: Four constants for initial positions and velocities of both particles.

$\vec{A}_+$ and $\vec{A}_-$ describe 2 independent modes.

$$\vec{A}_+ \cdot \vec{A}_- = 1 + \frac{1}{4}(1+\sqrt{5})(1-\sqrt{5}) = 1 + \frac{1}{4}(1-5) = 0.$$

Symmetric matrix must have orthogonal eigenvectors with diff. eigenvalues.

Driven coupled oscillators



$$mX_1'' = -kX_1 - k(X_1 - X_2) - bX_1' + F\cos\omega t$$

$$mX_2'' = -kX_2 - k(X_2 - X_1) - bX_2'$$

Textbook uses physics intuition to guess the solution. We will do it formally.

$$X_1'' + \gamma X_1' + \frac{k+k}{m} X_1 - \frac{k}{m} X_2 = f e^{i\omega t}$$

$$X_2'' + \gamma X_2' - \frac{k}{m} X_1 + \frac{k+k}{m} X_2 = 0$$

$$\Rightarrow \vec{X}'' + \gamma \vec{X}' + \begin{pmatrix} \frac{k+k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k+k}{m} \end{pmatrix} \vec{X} = f \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega t}$$

Approach 1: Ansatz $\vec{X} = \vec{A} e^{i\omega t}$ $\vec{X}' = i\omega \vec{A} e^{i\omega t}$, $\vec{X}'' = -\omega^2 \vec{A} e^{i\omega t}$

$$\Rightarrow \left[-\omega^2 + i\gamma\omega + \frac{1}{m} \begin{pmatrix} k+k & -k \\ -k & k+k \end{pmatrix} \right] \vec{A} = f \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\Omega & -k/m \\ -k/m & \Omega \end{pmatrix} \vec{A} = \begin{pmatrix} f \\ 0 \end{pmatrix} \quad \Omega = -\omega^2 + i\gamma\omega + (k+k)/m$$

$$-k/m A_1 + \Omega A_2 = 0 \Rightarrow A = \begin{pmatrix} -\Omega \\ k/m \end{pmatrix} n \Rightarrow (\Omega^2 - k^2/m^2) n = f$$

$$\Rightarrow A = \begin{pmatrix} -\Omega \\ k/m \end{pmatrix} \frac{f}{\Omega^2 - k^2/m^2}$$

$$\Rightarrow \vec{X} = \begin{pmatrix} -\Omega \\ k/m \end{pmatrix} \text{Re} \frac{e^{i\omega t}}{\Omega^2 - k^2/m^2} f$$

Approach 2: From observation (physics intuition) we can check the center of mass $X_1 + X_2$ and relative $X_1 - X_2$ motion.

$$X_1'' + \gamma X_1' + \frac{k+k}{m} X_1 - \frac{k}{m} X_2 = f e^{i\omega t}$$

$$X_2'' + \gamma X_2' - \frac{k}{m} X_1 + \frac{k+k}{m} X_2 = 0$$

$$\Rightarrow X \equiv (X_1 + X_2)/2$$

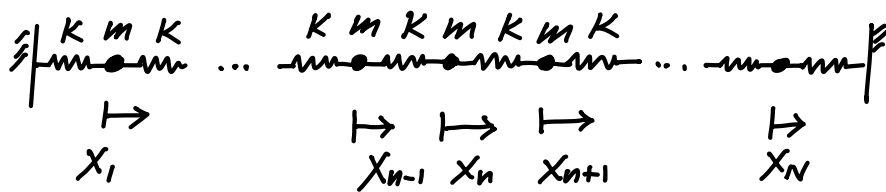
$$u \equiv X_1 - X_2$$

$$X'' + \gamma X' + \frac{k}{m} X = \frac{f}{2} e^{i\omega t}$$

$$u'' + \gamma u' + \frac{k+2k}{m} u = \frac{f}{2} e^{i\omega t}$$

$\Rightarrow$ Two uncoupled oscillators.

N -coupled oscillators.



X_i : deviation of the i th mass from the equilibrium position.

$$m x_n'' = \overbrace{F_{\text{Left}}}^{F_{\text{Left}}} + \overbrace{F_{\text{Right}}}^{F_{\text{Right}}} = -k(x_n - x_{n-1}) - k(x_n - x_{n+1})$$

$$\begin{aligned} x_n'' &= -2Kx_n + Kx_{n-1} + Kx_{n+1} \\ &\equiv -2\omega_0^2 x_n + \omega_0^2 x_{n-1} + \omega_0^2 x_{n+1} \\ &= \omega_0^2 (x_{n-1} - 2x_n + x_{n+1}) \end{aligned}$$

Boundary (imagine $x_0 = x_{N+1} = 0$ because of the wall(s))

$$\begin{aligned} x_1'' &= \omega_0^2 (-2x_1 + x_2) \\ x_2'' &= \omega_0^2 (x_1 - 2x_2 + x_3) \\ &\vdots \\ x_i'' &= \omega_0^2 (x_{i-1} - 2x_i + x_{i+1}) \\ &\vdots \\ x_N'' &= \omega_0^2 (x_{N-1} - 2x_N) \end{aligned} \Rightarrow \vec{x}'' = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \omega_0^2 \underbrace{\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -2 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -2 & 1 & 0 & \dots \\ \vdots & & & \vdots & & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & -2 \end{pmatrix}}_{\hat{M}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

$$\Rightarrow \vec{x}'' = \omega_0^2 \hat{M} \vec{x}$$

Standard approach: Ansatz $\vec{x} = \vec{A} e^{i\omega t} \Rightarrow \vec{x}' = i\omega \vec{A} e^{i\omega t}, \vec{x}'' = -\omega^2 \vec{A} e^{i\omega t}$

$$\Rightarrow -\omega^2 \vec{A} = \omega_0^2 \hat{M} \vec{A}$$

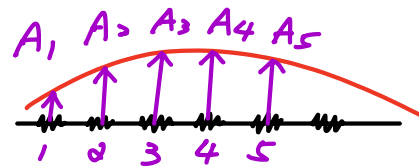
standard eigenvalue-eigenvector problem $\hat{M} \vec{A} = -\frac{\omega^2}{\omega_0^2} \vec{A} \equiv \lambda \vec{A}$
↑ eigenvalue

Math: given a $N \times N$ matrix
 there are N eigenvalues
 $2N$ eigenfreq. we need

$2N$ boundary conditions
 for positions & velocities of
 all N masses.

Smart guess of the solution

Given - $\frac{\omega^2}{\omega_0^2} A_n = A_{n-1} - 2A_n + A_{n+1}$



Ansatz $A_1 = \sin \epsilon$ $A_2 = \sin 2\epsilon$ $A_3 = \sin 3\epsilon$ $A_n = \sin n\epsilon$

$$\Rightarrow \left(2 - \frac{\omega^2}{\omega_0^2}\right) A_n = A_{n-1} + A_{n+1}$$

$$\Rightarrow 2 - \frac{\omega^2}{\omega_0^2} = \frac{A_{n+1} + A_{n-1}}{A_n} = \frac{\sin(n+1)\epsilon + \sin(n-1)\epsilon}{\sin n\epsilon} = \frac{2 \sin n\epsilon \cos \epsilon}{\sin n\epsilon} = 2 \cos \epsilon$$

$$\Rightarrow \omega^2 = \omega_0^2 (2(1 - \cos \epsilon)) = 4\omega_0^2 \sin^2 \epsilon / 2 \Rightarrow \omega = \pm 2\omega_0 \sin \epsilon / 2$$

What are the possible values for ϵ ?

From boundary conditions: $A_0 = A_{N+1} = 0$

$$\Rightarrow \sin(N+1)\epsilon = 0 \text{ or } (N+1)\epsilon = m\pi, \epsilon = m\pi/(N+1), \quad m = 0, 1, 2, 3, \dots, N, N+1$$

$$\Rightarrow \omega_m = \pm 2\omega_0 \sin \epsilon_m / 2$$

both lead to zero displacement

Sanity check: N eigenmodes, $2N$ eigenfreqs for N particles

General solution ($A_0 = A_{N+1} = 0$)

$$X_n(t) = \sum_{m=1}^N C_m \sin n\epsilon_m e^{-i\omega_m t} + D_m \sin n\epsilon_m e^{i\omega_m t}$$

$$= \sum_m \sin n\epsilon_m (E_m \cos \omega_m t + F_m \sin \omega_m t)$$

$\epsilon_m = m\pi/(N+1)$
 $\omega_m = 2\omega_0 \sin \epsilon_m$

Without the B.C show that the general solution is

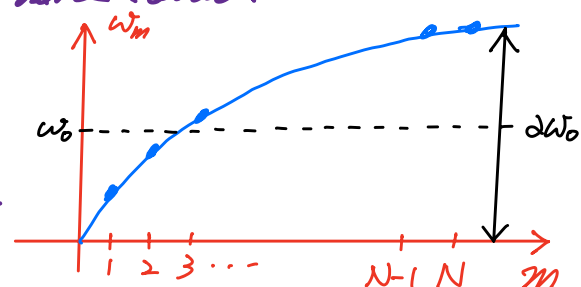
$$X_n(t) = \sum_{m=1}^N (G_m \cos n\epsilon_m + H_m \sin n\epsilon_m) (I_m \cos \omega_m t + J_m \sin \omega_m t)$$

$$= \sum_{m=1}^N K_m e^{i(n\epsilon_m + \omega_m t)} + L_m e^{i(-n\epsilon_m + \omega_m t)} + M_m e^{i(n\epsilon_m - \omega_m t)} + N_m e^{-i(n\epsilon_m + \omega_m t)}$$

Check textbook ansatz $A_n = \omega \sin n\epsilon$, which gives the same result.

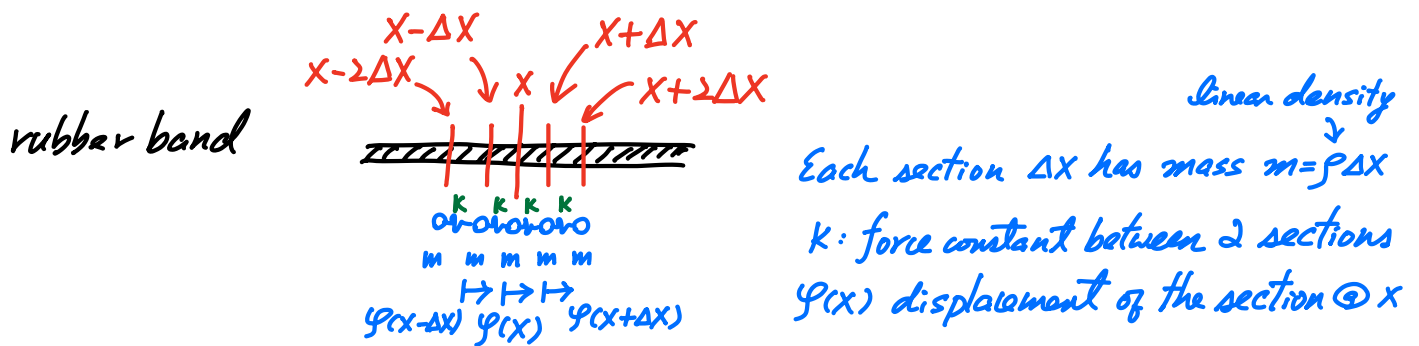
Complex ansatz $A_n = e^{i n\epsilon}$

$$\Rightarrow 2 - \frac{\omega^2}{\omega_0^2} = \frac{e^{i(n+1)\epsilon} + e^{i(n-1)\epsilon}}{e^{i n\epsilon}} = e^{i\epsilon} + e^{-i\epsilon} = 2 \cos \epsilon$$



Continuous mass-spring system: Wave eqn.

We model an elastic string as an infinite mass-spring chain



$$m \partial_t^2 \varphi(x, t) = K [\varphi(x - \Delta x) - 2\varphi(x) + \varphi(x + \Delta x)]$$

$$\rho \Delta x \partial_t^2 \varphi(x, t) = K \left[\frac{\varphi(x + \Delta x) - \varphi(x)}{\Delta x} - \frac{\varphi(x) - \varphi(x - \Delta x)}{\Delta x} \right] \Delta x$$

$$= K \left[\partial_x \varphi(x + \frac{\Delta x}{2}) - \partial_x \varphi(x - \frac{\Delta x}{2}) \right] \Delta x = K \partial_x^2 \varphi(x) \Delta x^2$$

$$\rho \partial_t^2 \varphi = K \Delta x \partial_x^2 \varphi \equiv E \partial_x^2 \varphi \quad E: \text{Young's modulus.}$$

Wave equation $\partial_t^2 \varphi = v^2 \partial_x^2 \varphi$ and order partial diff. eqn. $v^2 \equiv E/\rho$

Linearity: If φ_1 & φ_2 are solutions, so is $A\varphi_1 + B\varphi_2$ a solution.

Ansatz $\varphi(x, t) = A(x) e^{i\omega t}$

$$\Rightarrow -\omega^2 A e^{i\omega t} = v^2 A'' e^{i\omega t}$$

$$\Rightarrow A'' = -\frac{\omega^2}{v^2} A$$

Use ansatz again $A(x) \equiv e^{ikx}$, $A'' = -k^2 A$

$$\Rightarrow k = \pm \frac{\omega}{v}$$

$$-\omega^2 e^{ikx} = -v^2 k^2 e^{ikx}$$

$$\Rightarrow \omega^2 = v^2 k^2 \Rightarrow \omega = \pm vk$$

$$\Rightarrow \varphi = (A e^{ikx} + B e^{-ikx})(C e^{i\omega t} + D e^{-i\omega t})$$

$$= A' e^{i(kx + \omega t)} + B' e^{i(kx - \omega t)} + C' e^{i(-kx + \omega t)} + D' e^{i(-kx - \omega t)}$$

$$= A' e^{ik(x + vt)} + B' e^{ik(x - vt)} + C' e^{-ik(x - vt)} + D' e^{-ik(x + vt)}$$

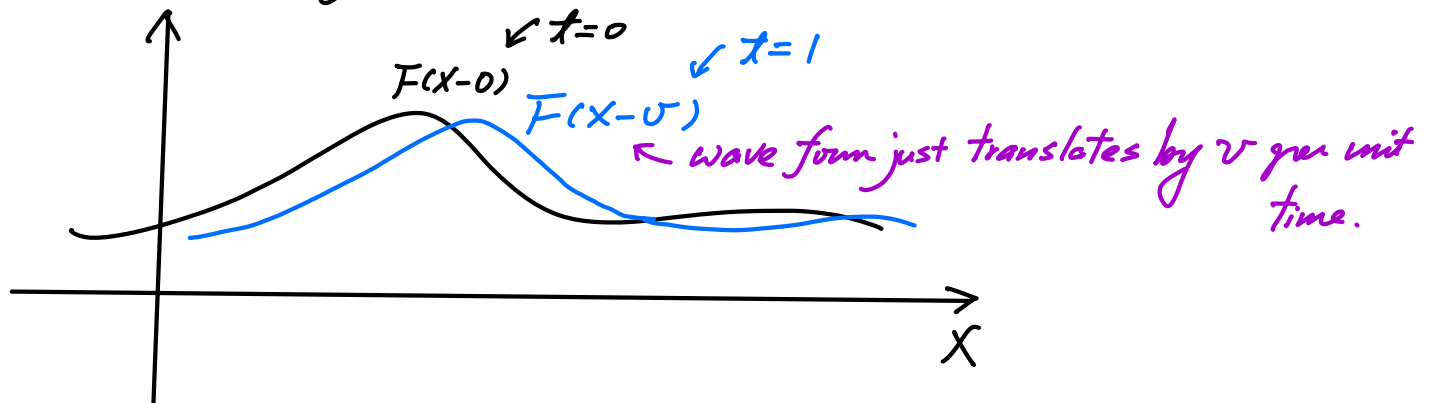
In fact any function $F(x \pm vt)$ are solutions of the wave eqn.
 u'''

Proof: Left side: $\partial_x F = \frac{\partial}{\partial x} F(u) = \frac{\partial u}{\partial x} \frac{dF}{du} = F'$. $\partial_x^2 F = F''$

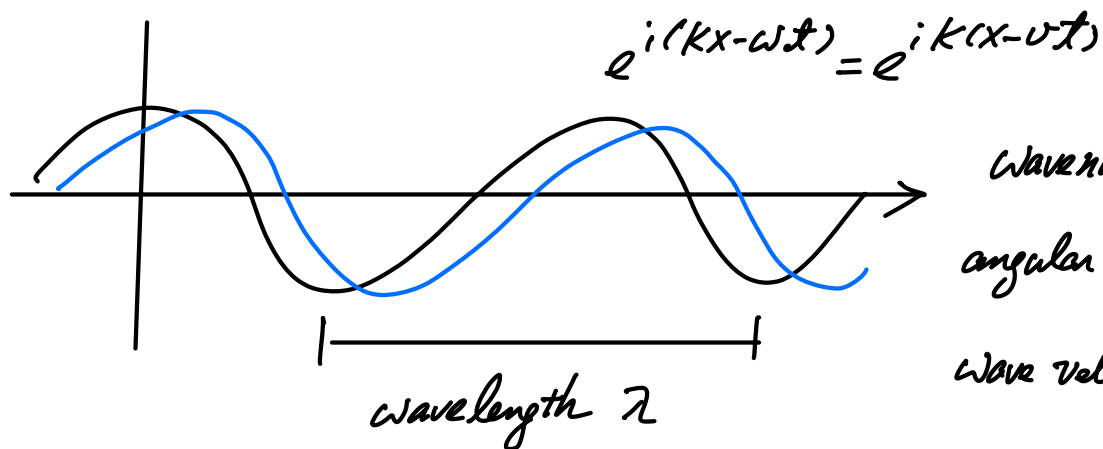
Right side: $\partial_t F = \frac{\partial u}{\partial t} \frac{dF}{du} = \pm v F'$. $\partial_t^2 F = v^2 F''$

$F(x-vt)$ is a travelling wave moving in the $+x$ direction

$F(x+vt)$ is a travelling wave moving in the $-x$ direction



For a sinusoidal wave



wavenumber $k = \frac{2\pi}{\lambda}$
 angular freq $\omega = kv$
 wave velocity $v = \frac{\omega}{k} = f\lambda$