

Fourier expansion and Fourier transform

How do we solve $\hat{L}x = f$ with arbitrary force f ?

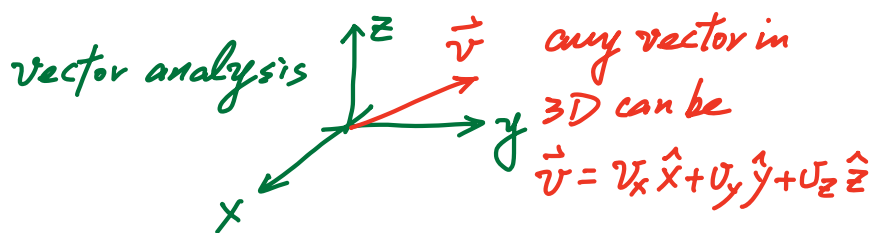
- single particle: $x'' + \gamma x' + \omega_0^2 x = f(t)$
- multi particles: $\ddot{\vec{x}} + \hat{\gamma} \dot{\vec{x}} + \hat{M} \vec{x} = \vec{f}(t)$
- Continuous system $\partial_t^2 \varphi - v^2 \partial_x^2 \varphi = f(x, t)$

Claim: Once you know how to solve $f(t) = e^{\alpha t}$, you know how to solve any function $f(t)$

Idea:

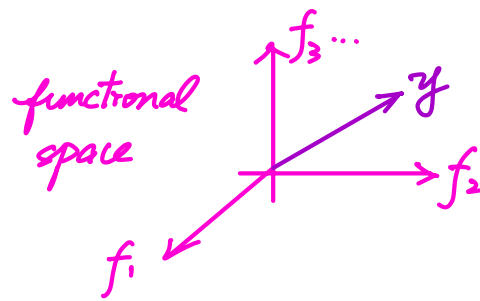
$$\begin{aligned}\hat{L}x_1 &= A_1 e^{\alpha_1 t} \\ \hat{L}x_2 &= A_2 e^{\alpha_2 t} \\ &\vdots \\ + \\ \hline \hat{L} \sum_i x_i &= \sum_i A_i e^{\alpha_i t} \equiv f(t)\end{aligned}$$

Math: Any continuous function $y(x)$ can be expanded by a set of complete set of $\cos \alpha x$, $\sin \alpha x$ or e^{ikx} .



$\hat{x}, \hat{y}, \hat{z}$ form a complete and orthogonal set of basis.

$$\begin{aligned}\hat{x} \cdot \hat{x} &= \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1 & \vec{v} &= v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \\ \hat{x} \cdot \hat{y} &= \hat{y} \cdot \hat{z} = \hat{x} \cdot \hat{z} = 0 & v_x &= \vec{v} \cdot \hat{x}, v_y = \vec{v} \cdot \hat{y}, v_z = \vec{v} \cdot \hat{z}\end{aligned}$$



Scalar product in functional space.

$\{f_i\}$ form a complete and orthogonal set of basis.

Orthornormal basis: $\int f_i^*(x) f_j(x) dx = \delta_{ij}$. $\delta_{ij} = 1$ when $i=j$
 $= 0$ when $i \neq j$

$y(x) = \sum_i A_i \underline{f_i(x)}$. Amplitude $A_i = \int f_i^*(x) y(x) dx$
 ampu. basis

Fourier Expansion for a periodic function $y(x+L) = y(x)$

$$\text{Claim: } y(x) = A_0 + \sum_n A_n \cos n \frac{2\pi x}{L} + \sum_n B_n \sin n \frac{2\pi x}{L}$$

The complete set here is $\{1, \cos n \frac{2\pi x}{L}, \sin n \frac{2\pi x}{L}\}$

These are all sin & cos with same period L .

$$\text{Check orthogonality: } \int_0^L 1 \cdot \cos n \frac{2\pi x}{L} dx = \int_0^L 1 \cdot \sin n \frac{2\pi x}{L} dx = \int_0^L \sin n \frac{2\pi x}{L} \cos m \frac{2\pi x}{L} dx = 0$$

$$\int_0^L 1 \cdot 1 dx = L. \int_0^L \sin n \frac{2\pi x}{L} \sin m \frac{2\pi x}{L} dx = \int_0^L \cos n \frac{2\pi x}{L} \cos m \frac{2\pi x}{L} dx = \frac{L}{2} \delta_{mn}$$

$$\Rightarrow y(x) = \boxed{\frac{1}{L} \int_0^L y(x) dx} + \boxed{\frac{2}{L} \sum_n \cos n \frac{2\pi x}{L} \int_0^L \cos n \frac{2\pi x}{L} y(x) dx} \quad \text{cos terms}$$

$$\quad \text{constant term} + \boxed{\frac{2}{L} \sum_n \sin n \frac{2\pi x}{L} \int_0^L \sin n \frac{2\pi x}{L} y(x) dx} \quad \text{sin terms}$$

Remark: The orthornormal basis is $\left\{ \frac{1}{\sqrt{L}}, \frac{\sqrt{2}}{\sqrt{L}} \cos n \frac{2\pi x}{L}, \frac{\sqrt{2}}{\sqrt{L}} \sin n \frac{2\pi x}{L} \right\}$

constant term is just the mean.

Alternatively, we can use exp function

$$y(x) = A_0 + \sum_n A_n (e^{-in2\pi x/L} + e^{in2\pi x/L}) + \sum_n B_n (e^{-in2\pi x/L} - e^{in2\pi x/L})$$

$$= \sum_{n=-\infty}^{\infty} C_n \underline{e^{in2\pi x/L}}$$

basis

$$\int_0^L e^{in2\pi x/L} e^{-im2\pi x/L} dx = L \delta_{nm}$$

Fourier transform. (general case of $y(x)$)

Now let $L \rightarrow \infty$.

$$y(x) = \int_{-\infty}^{\infty} \underbrace{y(k)}_{\text{coeff.}} \underbrace{e^{ikx}}_{\text{basis}} dk$$

$$\int e^{-ikx} y(x) dx$$

$$= \iint e^{-ikx} y(k') e^{ik'x} dk' dx$$

$$= \int y(k') \int e^{i(k'-k)x} dx dk'$$

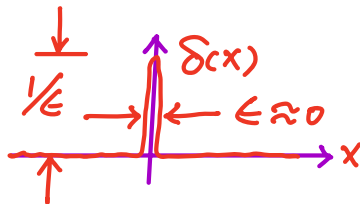
$$= 2\pi \int y(k') \delta(k'-k) dk'$$

$$= 2\pi y(k)$$

$$\Rightarrow y(k) = \frac{1}{2\pi} \int y(x) e^{-ikx} dx$$

$$\int_{-\infty}^{\infty} e^{-ik'x} e^{ikx} dx = 2\pi \delta(k-k')$$

Dirac's δ function: Unity area but infinitely narrow.



$$\begin{aligned} \delta(x) &= 0 \text{ for } x \neq 0 \\ \delta(x) &\rightarrow \infty \text{ for } x = 0 \\ \int \delta(x) dx &= 1 \end{aligned}$$

$$\int f(x) \delta(x) dx = f(0).$$

$$\begin{cases} y(x) = \int y(k) e^{ikx} dk & \leftarrow \text{Fourier transform} \\ y(k) = \frac{1}{2\pi} \int y(x) e^{-ikx} dx & \leftarrow \text{inverse Fourier transform} \end{cases}$$

$$\begin{cases} y(t) = \int y(\omega) e^{i\omega t} d\omega & \leftarrow \text{Fourier transform} \\ y(\omega) = \frac{1}{2\pi} \int y(t) e^{-i\omega t} dt & \leftarrow \text{inverse Fourier transform} \end{cases}$$

$y(\omega)$: temporal spectral function in freq domain $y(k)$: momentum space
 $y(t)$: temporal function in time domain $y(x)$: real space.