P452 Quiz 2

Your name:______________

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1 Fundamental Scattering Relations

The radial wavefunction $u(r)$ describes the relative motion of two colliding particles.

- **Scattering Amplitude (f):** $\psi(r) \sim e^{ikz} + f\frac{e^{ikr}}{r}$
- **S-Matrix (S):** $u(r) = r\psi(r) \sim e^{-ikr} - Se^{ikr}$, where $|S| = 1$
- **Phase Shift (δ):** $S = e^{2i\delta}$, where $\delta \in \mathbb{R}$ for elastic scattering
- **K-Matrix (K):** $u(r) \sim \sin(kr) + K\cos(kr)$, where $K = K^* = \tan\delta$

Low-Energy Limit ($k \to 0$)

In the ultracold regime, scattering is dominated by the s-wave channel.

- **Scattering Length (a):** $a = -\lim_{k\to 0}\frac{\tan\delta}{k} = -\lim_{k\to 0} f$
- **Total Cross Section (σ):** $\sigma = \frac{4\pi}{k^2}\sin^2\delta = \frac{4\pi a^2}{1+k^2a^2}$
- **Effective Range Expansion:** $k\cot\delta \approx -\frac{1}{a} + \frac{1}{2}r_e k^2$
- **Interaction Strength (g):** $g = \frac{4\pi\hbar^2 a}{m}$

2 Many-Body Hamiltonians & Models

These models are used to simulate complex phenomena in optical lattices.

- **Bose-Hubbard Model:** $H = -J\sum_{\langle i,j\rangle} b_i^\dagger b_j + \frac{U}{2}\sum_i n_i(n_i - 1)$
- **Heisenberg Spin Model:** $H = J\sum_{\langle i,j\rangle} \mathbf{S}_i \cdot \mathbf{S}_j$
- **Feshbach Resonance:** $a(B) = a_{bg}\left(1 - \frac{\Delta B}{B - B_0}\right)$

Instructions: Circle the answer for the following 12 questions in 20 minutes.

Part 1: Many-Body Physics & Simulation

1. For cold atoms in optical lattices, the effective coupling strength g is primarily determined by:
   (a) The depth of interaction potential
   (b) The scattering length a
   (c) The number of particles
   (d) The speed of light

2. The Hubbard model is widely used in research to simulate:
   (a) Neutron stars
   (b) High-Tc superconductors
   (c) Bose-Fermi mixtures in an optical dipole trap
   (d) Molecular potential of alkali atoms

3. In the Bose-Hubbard model, the "Mott Insulator" phase occurs when:
   (a) Tunneling J dominates over interaction U
   (b) Interaction U dominates over thermal energy $k_B T$
   (c) The system becomes highly incompressible
   (d) The scattering length is small and negative

4. In the numerical simulation of many-body systems, the size of the Hilbert space $\mathcal{H}$ is determined by particle statistics and the number of available states. For a system consisting of two sites ($M = 2$) and two particles ($N = 2$), which of the following statements correctly identifies the dimension of the Hilbert space?
   (a) For **spinless bosons**, the dimension is $\dim(\mathcal{H}) = 4$, as each particle can independently occupy either site.
   (b) For **spinless fermions**, the dimension is $\dim(\mathcal{H}) = 1$, because the Pauli Exclusion Principle restricts the system to a single configuration where each site is occupied by exactly one particle.
   (c) For **distinguishable spin-1/2 particles** (with one particle fixed per site), the dimension is $\dim(\mathcal{H}) = 2$, representing the two possible alignments of the total spin.
   (d) For a **Bose-Fermi mixture** (one spinless boson and one spinless fermion), the dimension is $\dim(\mathcal{H}) = 6$, representing the sum of the individual particle dimensions.

5. The Gross-Pitaevskii equation is frequently used to describe the mean-field dynamics of a Bose-Einstein condensate (BEC). In the following equation, what does the nonlinear term $g|\psi|^2$ physically represent?

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + g|\psi|^2\right)\psi$$

   (a) The interaction energy of two atoms in a lattice site.
   (b) A mean-field interaction that stablizes BEC when scattering length is $a > 0$.
   (c) The external trapping potential mediated by the other atoms.
   (d) It is negligible under Thomas-Fermi approximation.

6. For three antiferromagnetically coupled Ising spins ($s = 1/2$) arranged in a triangle, what is the degeneracy (number of states) in the ground state manifold?

   (a) 2
   (b) 4
   (c) 6
   (d) 8

7. You have utilized **QuSpin** to solve many-body Hamiltonians and **Qiskit** to simulate quantum circuits. Which of the following statements regarding computational complexity is correct?

   (a) Mapping a many-body system to a digital quantum circuit always requires fewer qubits than the number of physical atoms being simulated.
   (b) Simulating a 6×6 Heisenberg spin model is quite feasible on a laptop, but an 8×8 system is 2^{28} times harder due to the exponential growth of the Hilbert space.
   (c) Utilizing symmetry sectors, such as fixed total magnetization ($S_z = 0$) or translational symmetry, helps reduce the effective dimension of the Hilbert space.
   (d) For the same number of sites and particles, Exact Diagonalization with QuSpin is computationally harder for the Fermi-Hubbard model than the Bose-Hubbard model.

Part 2: Low Energy Quantum Scattering

8. **Find the correct statement on collisions of cold atoms.**

(a) Attractive potential leads to negative scattering phase shift $\delta < 0$.
(b) The total scattering cross section σ increases for atoms with greater kinetic energy.
(c) At low temperatures, collision cross section depends on the absolute value of the scattering length.
(d) Two Fermions in an optical tweezer, in the same spin state, but different motional vibrational states. They can still interact by $s-$wave scattering.

9. For an attractive "square-well" potential with radius R, the s-wave scattering length a

   (a) must be negative
   (b) is given by $-2R \leq a \leq 2R$
   (c) can be positive or negative or zero
   (d) is always given by the zero-crossing of the wavefunction $\psi(r = a) = 0$.

10. The S-matrix element $S = e^{2i\delta}$ satisfies the property of unitarity ($|S| = 1$). Physically, this represents:

    (a) Conservation of energy
    (b) Conservation of particle flux (probability)
    (c) Conservation of elastic and inelastic cross sections
    (d) The presence of interparticle

11. Prof. Zwierlein mentioned in his talk that neutron scattering length is -18.9 fm. Given that neutrons are fermions, what we can conclude?

    (a) Two neutrons in different state attracts.
    (b) Two neutrons in the same spin state repel.
    (c) Low energy neutrons cannot interact via the s-wave channel.
    (d) Two neutrons can form a bound state.

12. Near a Feshbach resonance, the scattering length $a(B)$ diverges when the magnetic field B reaches B_0. This occurs because:

    (a) The kinetic energy is converted into potential energy due to energy conservation.
    (b) A bound state in the closed channel aligns with the threshold of the open channel.
    (c) The scattering matrix diverges when the scattering phase shift is $\delta = 0, \pm\pi/2, \pm\pi....$
    (d) Ultracold atoms always attract each other.

Correct Answers & Explanations

Answer Key and Explanations

Part 1: Many-Body Physics Simulation

1. **Answer: (b)**

    - **Explanation:** In the low-energy limit of cold atoms, the interaction is characterized by s-wave scattering. The effective coupling strength g is defined as $g = \frac{4\pi\hbar^2 a}{m}$, where a is the s-wave scattering length.

2. **Answer: (b)**

    - **Explanation:** The Hubbard model is a pillar of condensed matter (e.g., high-T_c superconductors).

3. **Answer: (c)**

    - **Explanation:** The Mott Insulator phase occurs in the limit where on-site interaction U dominates over hopping J. This results in a fixed integer number of particles per site, making the system highly incompressible.

4. **Answer: (b)**

    - **Explanation:** For spinless fermions, the Pauli Exclusion Principle forbids multiple occupancy. For $N = 2$ particles in $M = 2$ sites, the Hilbert space dimension is $\binom{M}{N} = \binom{2}{2} = 1$.

5. **Answer: (b)**

    - **Explanation:** The mean-field interaction g is proportional to the s-wave scattering length a. Repulsive force between atoms stables the BEC.

6. **Answer: (c)**

    - **Explanation:** Geometrical frustration in an antiferromagnetic triangle of three Ising spins leads to 6 degenerate ground states: $\{\uparrow\uparrow\downarrow, \uparrow\downarrow\uparrow, \downarrow\uparrow\uparrow, \downarrow\downarrow\uparrow, \downarrow\uparrow\downarrow, \uparrow\downarrow\downarrow\}$.

7. **Answer: (c)**

    - **Explanation:** Numerical tools like QuSpin and DMRG exploit symmetry sectors (e.g., S_z conservation or translational symmetry) to block-diagonalize the Hamiltonian and reduce the computational dimension.

Part 2: Low Energy Quantum Scattering

8. **Answer: (c)**

   - **Explanation:** In the ultra-cold limit ($k \to 0$), the s-wave scattering cross-section for non-identical particles (or bosons) becomes $\sigma = 4\pi a^2$, which depends solely on the absolute value of the scattering length.

9. **Answer: (c)**

   - **Explanation:** For an attractive square well, the scattering length a is not fixed in sign; it cycles between positive and negative values as the well depth changes and supports more bound states.

10. **Answer: (b)**

    - **Explanation:** Unitarity ($|S| = 1$) is a mathematical requirement that ensures the conservation of particle flux, meaning no particles are created or destroyed during the elastic collision process.

11. **Answer: (a)**

    - **Explanation:** A large negative scattering length ($a \approx -18.9$ fm) indicates a strong attraction. Identical fermions in the same state cannot undergo s-wave scattering, so this interaction applies to neutrons in different states.

12. **Answer: (b)**

    - **Explanation:** A Feshbach resonance occurs when the energy of a bound state in a closed molecular channel is tuned to coincide with the zero-energy scattering threshold of the open channel.