Lecture 15 Wk 10 03/07 W Scale-invariance and renormalization group

What is scale invariance?
Physics: Phenomena that happen at all scales…
Math: $F(\lambda x) = \lambda^{\alpha} F(x)$
Examples: Fractals, snow flakes, icicles, sand piles, turbulence (?), stock market (?), American household income (?)…

Renormalization
Idea of block spin (Leo Kadanoff, 1966): physics is scale invariant near a critical point

Consider 1D Ising model (no B field): $G_B = -J \sum_{<i,j>} \sigma_i \sigma_j$

$Z(j,N) = \sum_{\sigma} e^{j(\sigma_1\sigma_2+\sigma_2\sigma_3)} e^{j(\sigma_3\sigma_4+\sigma_4\sigma_5)} \ldots$, where $j = \beta J$.

Now we sum over σ_2, σ_4 … and get

$$\begin{aligned} Z(j,N) &= \sum 2\cosh j(\sigma_1+\sigma_3) 2\cosh j(\sigma_3+\sigma_5)\ldots \\ &= \sum f(j) e^{j_1\sigma_1\sigma_3} f(j) e^{j_1\sigma_3\sigma_5}\ldots \\ &= f(j)^{N/2} Z(j_1, N/2) \end{aligned}$$

Here we can determine f and j' from

$\sigma_1 = \sigma_3 = -1$: $fe^{j_1} = 2\cosh 2j$ (same for $\sigma_1 = \sigma_3 = -1$)

$\sigma_1 = -\sigma_3 = 1$: $fe^{-j_1} = 2$ (same for $\sigma_1 = -\sigma_3 = -1$)

So we have $f = 2\sqrt{\cosh 2j}$ and $j_1 = \frac{1}{2}\ln\cosh(2j)$

We may iterate the process m times and get

$Z(j,N) = f(j)^{N/2} f(j_1)^{N/4} \ldots f(j_{m-1})^{N/2^m} Z(j_m, N/2^m)$, and

$j_m = \frac{1}{2}\ln\cosh(\ln\cosh(\ln\cosh\ldots)\ldots))$

The iterative evolution of j is called RG flow and in this example, $\lim_{m\to\infty} j_m = 0$. Thus

$\lim_{m\to\infty} f(j_m) = 2$, and $\lim_{m\to\infty} Z(j_m, N/2^m) = Z(0, N/2^m) = 2^{N/2^m}$

Free energy per particle $g(j) = \frac{G}{N} = -\frac{kT}{N}\ln Z$ follows

$$\frac{g(j)}{kT} = -\frac{j_1+\ln 2}{2} + \frac{1}{2}\frac{g(j_1)}{kT} = -\ln 2 - \frac{j_1}{2} - \frac{j_2}{2^2} - \ldots$$

1D Ising chain has no fixed point so RG flow has fixed points at $\lim_{m\to\infty} j_m = 0, \infty$.(reverse flow)

In models with a phase transition, scale invariance suggests that critical points would be the fixed points of RG flow.

When the system function (partition, Hamiltonian…) can be reduced in size iteratively with modified parameters, the system is called renormalizable. The evolution of the parameters is the RG flow and fixed points in the flow indicate the emergence of scale invariance.