When discrete, equally spaced data is given as \( x_0, x_1, \ldots, x_{N-1} \)

Fast Fourier transform is defined as

\[
K_j = \sum_{k=0}^{N-1} x_k e^{-i2\pi jk/N}, \quad j = 0 \ldots N-1
\]

Inverse FFT is

\[
x_k = \frac{1}{N} \sum_{j=0}^{N-1} K_j e^{i2\pi jk/N}, \quad l = 0 \ldots N-1
\]

This equation defines the meaning of \( K_j \):

\[
\frac{K_j}{N} \text{ is the coefficient of the term } e^{i2\pi jk/N} = e^{i\frac{2\pi j}{N}} \text{ whose freq is } 2\pi j/N
\]

If the step size of \( x_e \) is \( \Delta x \), the period is \( \frac{N}{j} \Delta x \), freq \( \frac{2\pi j}{N\Delta x} \)

**Application:** Given a discrete sample of \( f_e = f(x) \), \( l = 0 \ldots N-1 \)

What do we know about its spectrum \( f(w) = \frac{1}{\pi} \int f(x) e^{-i\omega t} dt \)

from the discrete FT \( A_j = \sum_{k=0}^{N-1} f_k e^{-i\omega jk/N} \)?

\[
f_e = \int f(w) e^{i\omega t} dt
\]

\[
f_e = \int f(w) e^{i\omega t} dt
\]

\[
A_j = \sum_{k=0}^{N-1} f_k e^{-i\omega jk/N} \text{ dw}
\]

\[
= \int f(w) \sum_{k=0}^{N-1} e^{i\omega t} e^{-i\omega jk/N} \text{ dw}
\]

\[
= \int f(w) \frac{1 - e^{i\alpha}}{1 - e^{i\omega}} \text{ dw}
\]

\[= \int f(w) g(wj - w) \text{ dw}
\]

\[
= \int f(w) e^{i(N-1)\alpha/2} \frac{\sin \alpha/2}{\sin \omega j/2} \text{ dw}
\]

\[
= \int f(w) g(wj - w) \text{ dw}
\]

\[
\text{FWHM} = \frac{\Delta \omega}{N} \Rightarrow \Delta \omega = \frac{\omega j}{N\Delta x}
\]
Assume \( f(w) \) is a slow-varying function

\[
A_j = \frac{2\pi}{\Delta t} \sum_{k=0}^{N-1} f(w_j + 2\pi k/\Delta t)
\]

For \( 0 \leq w_j \leq \pi/\Delta t \)

\[
A_j \approx \frac{2\pi}{\Delta t} \left[ f(w_j) + f(w_j - 2\pi/\Delta t) \right] = \frac{2\pi}{\Delta t} \left[ f(w_j) + f^*(w_{N-j}) \right]
\]

(For real \( f \)) \( \Rightarrow A_{wj} = \frac{2\pi}{\Delta t} \left[ f(w_{N-j}) + f^*(w_j) \right] \)

\( A_j = A_{wj} \)

**Conclusion**

Given a discrete data set of length \( N \): \( f_x = f(x = \Delta x) \), \( \Delta x = 0 \ldots N-1 \)

DFT or FFT gives

\[
A_j = \frac{N}{\Delta t} \sum_{k=0}^{N-1} f_x e^{-i2\pi jk/N}
\]

\[
= \int f(w) e^{-i(N-1)\Delta x \Delta t} \frac{\sin N\Delta t}{\sin \Delta t} d\omega
\]

\[
\approx \frac{2\pi}{\Delta t} \left[ f(w_j) + f^*(w_{N-j}) \right]
\]

\( w_j = 2\pi j/N\Delta t \)