Expanding Mathematical Models for N-Stage Theoretical Quantum Transduction

Evelyn Voss[1, *] and Chiao-Hsuan Wang[2, 3, 4]
(UCTS Exchange Program)
[1] *Department of Physics, University of Chicago, Chicago, Illinois 60637*
[2] *Department of Physics and Center for Theoretical Physics, National Taiwan University, Taipei 106319, Taiwan*
[3] *Center for Quantum Science and Engineering, National Taiwan University, Taipei 106319, Taiwan*
[4] *Physics Division, National Center for Theoretical Sciences, Taipei, 106319, Taiwan*
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With an increasing demand for the advantages that quantum science brings to computation, communication, and more, the demand for the ability to transfer quantum information between media also rises. In this paper, we expand the available constraint formulas describing theoretical quantum transduction to include N-Stage Mode Transduction. We will discuss quantum transduction with its current theorized and hypothesized benefits and challenges, the mathematical models currently used to understand it, and the 0-Stage Bosonic Mode Transduction case. The formalism for quantum transduction used in this paper is applicable to many linear conversions between microwave, optical, or a combination of these two systems. Assuming intermediate modes are lossless, resonance across all modes, and the optimal reflectionless condition, we present an equation for the external loss rates of the end modes.

I. INTRODUCTION

A. Motivations and Background

There is still much room for theoretical advancement in quantum information science that will aid the effective implementation of quantum devices and systems. Quantum transduction presents a method for a transfer of information that is crucial to effectively harness the full benefits of quantum information systems.

The conversion between microwave and optical frequencies has been a large area of interest in developing quantum transduction. Successful transduction would combine the long-range communication abilities of optical fibers [1] with computation abilities of superconducting processors [2] [3]. In this section, we present a brief overview of the derivation for the quantum transduction formalism used in this paper.

1. *Continuous-Variable Quantum Systems*

In quantum information science, there are two types of quantum states, discrete-variable and continuous-variable, that each represent a type of quantum information. Discrete-variable quantum information science is used, for example, in quantum computing, where the qubit has two distinguishable states as a complete basis. Continuous-variable quantum systems operate in an infinite-dimensional Hilbert Space described by observables with continuous eigenspectra. More simply put, they are needed to describe variables with continuous measurable quantities, such as position or momentum. Importantly, these systems can describe the propagating electromagnetic field and other bosonic systems. Using this theory allows us to understand much about the physical modeling of quantum computation devices and their inner workings. In this paper, we consider an intermediary bosonic system capable of transferring quantum information from one system to another.

The primary tools for analyzing continuous-variable quantum information processing are Gaussian states and transformations. Gaussian states are quantum continuous-variable states that can be represented in terms of Gaussian functions, and Gaussian transformations take Gaussian states to Gaussian states. [4]

2. *Bosonic Systems*

All physical information in an N-mode bosonic system is contained within its quantum state, represented by its corresponding density matrix, $\hat{\rho}$. Every system can be characterized by its Hamiltonian, $\hat{H}$, which acts on a quantum state and describes how the state evolves over time. The Hamiltonian operator is itself comprised of operators that each are correlated with an observable (something about the state that can be measured) in the system. All operators will be denoted with the hat symbol (e.g. $\hat{H}$, $\hat{m}_j$, $\hat{a}$, $\hat{b}$).

3. *Quantum Transduction*

Classical transduction is the process by which different classical signals can be translated from one physical

* eavoss@uchicago.edu

medium to another. For example, a classical transducer converts sound waves to electrical signals in a radio system that can then be more easily transmitted across long distances. Similarly, there advantages in converting types of quantum information across different quantum carrying media. As the demand for more robust quantum systems and usage of quantum information grows, quantum transducers will become essential in realizing quantum networks. For example, this technology acts as the crucial medium by which information from local quantum processors is converted into forms suited for long-range quantum communication. Much work has been done to establish the theoretical framework for faithful microwave to optical quantum transduction (see [5] and [6]).

4. Transducers as Bosonic Systems

Our transducers can be thought of as multi-stage chains of bosonic modes with two end modes coupled externally. The interaction is quantified for each mode, $\hat{m}_j$, through coupling constants, g_j. For N-stage quantum transduction, we can represent the system with the Hamiltonian shown below in Eq. 1.

$$\hat{H} = -\sum_{i=1}^{N+2} \hbar\Delta_i \hat{m}_i^\dagger + \sum_{i=1}^{N+2} \hbar g_i (\hat{m}_{i+1}^\dagger \hat{m}_i + \hat{m}_i^\dagger \hat{m}_{i+1}) \quad (1)$$

The first portion of the sum represents the evolution of each individual mode, and the second portion of the sum describes the interaction between each of the modes [3].

II. METHODS

Our goal is to derive generalized equations describing N-stage quantum transduction to further understanding of the dissipation rate conditions for maximum efficiency transduction. This is important for understanding and working towards the physical implementation of quantum transducers.

A. Scattering Matrix

We can understand how quantum information is transformed as it travels through a transducer through the Scattering Matrix Formalism. For this paper, we will be considering a bidirectional transducer, where information flows from the first mode to the second mode and vice versa. This is known as Duplex Quantum Transduction. We will be following the calculation conventions and methods used in [7].

Thus, we have the $2N$ by $2N$ scattering matrix, $\boldsymbol{S}$, which is a Gaussian unitary transformation describing N-stage transduction. Below in Eq. (3) is a scattering matrix representing 0-stage transduction, i.e. transduction with no intermediary modes.

For simplicity, we consider the case in which the modes are on resonance and the information loss rates across the two modes are equal (i.e. detunings of modes $\hat{a}$ and $\hat{b}$ are $\Delta_a = \Delta_b = 0$ and information loss rates $\kappa_{e,a} = \kappa_{e,b} = \kappa_e$ and $\kappa_{i,a} = \kappa_{i,b} = \kappa_i$).

$$\begin{pmatrix} \hat{a}_{\text{out}} \\ \hat{b}_{\text{out}} \end{pmatrix} = \boldsymbol{S} \begin{pmatrix} \hat{a}_{\text{in}} \\ \hat{b}_{\text{in}} \end{pmatrix} \quad (2)$$

Above in Eq. 2 is the input-output formalism represented in terms of the scattering matrix where $\hat{a}_{\text{out}}$, $\hat{b}_{\text{out}}$ and $\hat{a}_{\text{in}}$, $\hat{b}_{\text{in}}$ are the respective output and input operators for modes $\hat{a}$ and $\hat{b}$.

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S_{11}} & \boldsymbol{S_{12}} \\ \boldsymbol{S_{21}} & \boldsymbol{S_{22}} \end{pmatrix} \quad (3)$$

Where

$$\boldsymbol{S_{11}} = \begin{pmatrix} \frac{4g^2-\kappa_e^2+\kappa_i^2}{4g^2+(\kappa_e+\kappa_i)^2} & \frac{4ig\kappa_e}{4g^2+(\kappa_e+\kappa_i)^2} \\ \frac{4ig\kappa_e}{4g^2+(\kappa_e+\kappa_i)^2} & \frac{4g^2-\kappa_e^2+\kappa_i^2}{4g^2+(\kappa_e+\kappa_i)^2} \end{pmatrix} \quad (4)$$

$$\boldsymbol{S_{12}} = \begin{pmatrix} \frac{4g^2-\kappa_e^2+\kappa_i^2}{4g^2+(\kappa_e+\kappa_i)^2} & \frac{4ig\kappa_e}{4g^2+(\kappa_e+\kappa_i)^2} \\ \frac{4ig\kappa_e}{4g^2+(\kappa_e+\kappa_i)^2} & \frac{4g^2-\kappa_e^2+\kappa_i^2}{4g^2+(\kappa_e+\kappa_i)^2} \end{pmatrix} \quad (5)$$

$$\boldsymbol{S_{21}} = \begin{pmatrix} \frac{4g^2-\kappa_e^2+\kappa_i^2}{4g^2+(\kappa_e+\kappa_i)^2} & \frac{4ig\kappa_e}{4g^2+(\kappa_e+\kappa_i)^2} \\ \frac{4ig\kappa_e}{4g^2+(\kappa_e+\kappa_i)^2} & \frac{4g^2-\kappa_e^2+\kappa_i^2}{4g^2+(\kappa_e+\kappa_i)^2} \end{pmatrix} \quad (6)$$

$$\boldsymbol{S_{22}} = \begin{pmatrix} \frac{4g^2-\kappa_e^2+\kappa_i^2}{4g^2+(\kappa_e+\kappa_i)^2} & \frac{4ig\kappa_e}{4g^2+(\kappa_e+\kappa_i)^2} \\ \frac{4ig\kappa_e}{4g^2+(\kappa_e+\kappa_i)^2} & \frac{4g^2-\kappa_e^2+\kappa_i^2}{4g^2+(\kappa_e+\kappa_i)^2} \end{pmatrix} \quad (7)$$

and

1. S_{ij} describes the interactions between modes i and j,
2. g is the coupling constant between the two modes,
3. κ_e is the external loss rate in both modes,
4. κ_i is the internal loss rate in both modes.

Figure 1 represents a visualization of 0-stage transduction with modes $\hat{a}$ and $\hat{b}$.

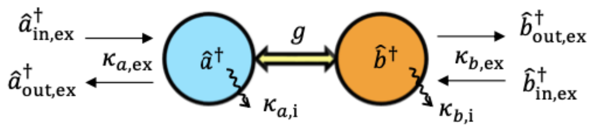

FIG. 1. Visualization of 0-stage transduction [3].

1. *Solving for Reflectionless Conditions using the Scattering Matrix*

Using the scattering matrix, we can solve for the reflection and transmission coefficients, which represent the proportion of information respectively reflected and transmitted at each mode. The formula for the coefficients is given below in Eq. 8 and Eq. 9 [7].

$$R_i = |S_{ii}|^2 \tag{8}$$

$$T_{ij} = |S_{ij}|^2 \tag{9}$$

$$\boldsymbol{S_{ij}} = \begin{pmatrix} \text{Re}\{S_{ij}\} & \text{Im}\{S_{ij}\} \\ \text{Im}\{S_{ij}\} & \text{Re}\{S_{ij}\} \end{pmatrix} = |S_{ij}|\boldsymbol{R}(\theta_{ij}). \tag{10}$$

Solving for the maximum transmission coefficient requires a reflectionless condition, that is $R_1 = R_2 = 0$. Using equations 4-7, 8-9, we attain conditions for the loss rates as seen in Eq. 11 below.

$$\Delta_1 = \frac{\kappa_e - \kappa_i}{\kappa_e + \kappa_i}\Delta_2 \tag{11}$$

$$\Delta_2 = \sqrt{\frac{(\kappa_e + \kappa_i)(4g^2 - \kappa_e^2 + \kappa_i^2)}{4(\kappa_e + \kappa_i)}} \tag{12}$$

While the scattering matrix formalism is helpful for explicitly solving any transmission or reflection coefficient, the notation becomes overwhelming as we look at multiple stage transduction. For example, the initial calculations for bireflectionless $N = 1$ transduction are shown in **Appendix A**. There is another more efficient method to find loss rates for the bireflectionless condition.

B. Generalized Matching Condition

Using the system Hamiltonian, Wang et al. (2022) [3] derives an explicit expression representing the efficiency, η, of a transducer (Eq. 13). This quantity describes the proportion of information transmitted from the input signal to the output signal. In 0-stage transduction, the efficiency is equivalent to the transmission coefficient between the two modes $T_{12} = T_{21}$. In perfect quantum transduction, we achieve unity efficiency $\eta = 1$, where no information is lost. This efficiency is dependent on the signal frequency ω. We will assume the work in a rotating frame such that $\omega = 0$ for simplicity.

$$\eta[\omega] \equiv \left|\boldsymbol{S}_{N\hat{b}^\dagger_{\text{out,ex}}\hat{a}^\dagger_{\text{in,ex}}}\right| \tag{13}$$

$$\eta[\omega] \equiv \left|\frac{\hat{b}^\dagger_{\text{out,ex}}}{\hat{a}^\dagger_{\text{in,ex}}}\right|^2 = \left|\frac{\hat{a}^\dagger_{\text{out,ex}}}{\hat{b}^\dagger_{\text{in,ex}}}\right|^2 \tag{14}$$

The input-output formalism that gives rise to the system Hamiltonian and mode operator equations results in Eq. 14. A more detailed breakdown of the derivation of the efficiency equation can be found in reference [3] and its supplemental material.

The N-Stage Conversion Efficiency is as follows in Eq. 15. Figure 2 shows a visualization of N-stage transduction with external modes $\hat{a}$, $\hat{b}$, and intermediate modes $\hat{m}_2$, ..., $\hat{m}_{N+1}$.

$$\eta_N(\omega) = \left|\frac{\sqrt{\kappa_{a,\text{ex}}\kappa_{b,\text{ex}}\prod_{j=1}^{N+1} g_j}}{\prod_{j=1}^{N+2}\chi_{j,\text{eff}}^{-1}}\right|^2 \tag{15}$$

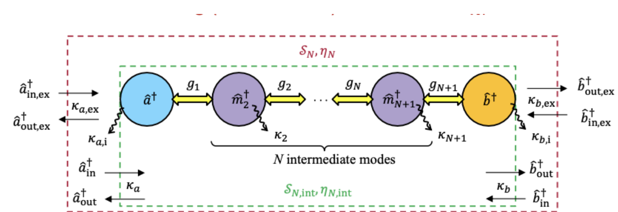

FIG. 2. Visualization of N-stage transduction [3].

The susceptibility of a mode, as defined in Eq. 16 below, quantifies the amount of information transferred between two modes in terms of the dissipation rate, detuning, and coupling constant.

$$\chi_j = \frac{1}{i(\omega + \Delta_j) + \kappa_j/2} \tag{16}$$

The effective mode susceptibility, $\chi_{j,\text{eff}}$, is defined in Eq. 17 below. It incorporates the susceptibilities from the mode j and all the couplings following it to the right, $g_j, g_{j+1}, \cdots, g_b$. Note that the transduction process is reciprocal, with identical conversion efficiency from either end to the other.

$$\chi_{j,\text{eff}}^{-1} \equiv \chi_j^{-1} + \frac{g_j^2}{\chi_{j+1}^{-1} + \frac{g_{j+1}^2}{\cdots + \frac{g_N^2}{\chi_{N+1}^{-1} + \frac{g_{N+1}^2}{\chi_{N+2}^{-1}}}}} \tag{17}$$

Wang et al. (2022) find conditions for maximum internal and external efficiency. We focus on overall efficiency, considering only the case where all modes are on resonance and there is no internal dissipation (i.e. $\kappa_2 = \kappa_3 = ... = \kappa_{N+1} = 0$). We use their equation for overall external efficiency (Eq. 18) to derive the results in the next portion of the paper.

$$(\chi_{a,\mathrm{ex}}^{-1})^* = g_1^2 \chi_{2,\mathrm{eff}} + \frac{\kappa_{a,i}}{2} \tag{18}$$

III. RESULTS

N Odd	$\kappa_{a,\mathrm{ex}}^2 = \left(\frac{\prod_{i=1}^{\frac{N+1}{2}} g_{2i-1}^2 \kappa_b}{\prod_{j=1}^{\frac{N+1}{2}} g_{2j}^2} + \kappa_{a,i} \right)^2$	$\kappa_{b,\mathrm{ex}}^2 = \left(\frac{\prod_{i=1}^{\frac{N+1}{2}} g_{2i-1}^2 \kappa_a}{\prod_{j=1}^{\frac{N+1}{2}} g_{2j}^2} + \kappa_{b,i} \right)^2$
N Even	$\kappa_{a,\mathrm{ex}}^2 = \left(\frac{4\prod_{i=1}^{\frac{N}{2}+1} g_{2i-1}^2}{\prod_{j=1}^{\frac{N}{2}} g_{2j}^2 \kappa_b} + \kappa_{a,i} \right)^2$	$\kappa_{b,\mathrm{ex}}^2 = \left(\frac{4\prod_{i=1}^{\frac{N}{2}+1} g_{2i-1}^2}{\prod_{j=1}^{\frac{N}{2}} g_{2j}^2 \kappa_a} + \kappa_{b,i} \right)^2$

TABLE I. Reflectionless Condition for N-Stage Transduction with Lossless Intermediate Modes and Resonance

N-Mode	$\kappa_{a,\mathrm{ex}}^2$	$\kappa_{b,\mathrm{ex}}^2$
$N = 1$	$\left(\frac{g_1^2\kappa_b}{g_2^2} + \kappa_{a,i}\right)^2$	$\left(\frac{g_1^2\kappa_a}{g_2^2} + \kappa_{b,i}\right)^2$
$N = 2$	$\left(\frac{4g_1^2 g_3^2}{g_2^2\kappa_b} + \kappa_{a,i}\right)^2$	$\left(\frac{4g_1^2 g_3^2}{g_2^2\kappa_a} + \kappa_{b,i}\right)^2$
$N = 3$	$\left(\frac{g_1^2 g_3^2\kappa_b}{g_2^2 g_4^2} + \kappa_{a,i}\right)^2$	$\left(\frac{g_1^2 g_3^2\kappa_a}{g_2^2 g_4^2} + \kappa_{b,i}\right)^2$
$N = 4$	$\left(\frac{4g_1^2 g_3^2 g_5^2}{g_2^2 g_4^2\kappa_b} + \kappa_{a,i}\right)^2$	$\left(\frac{4g_1^2 g_3^2 g_5^2}{g_2^2 g_4^2\kappa_a} + \kappa_{b,i}\right)^2$
$N = 5$	$\left(\frac{g_1^2 g_3^2 g_5^2\kappa_b}{g_2^2 g_4^2 g_6^2} + \kappa_{a,i}\right)^2$	$\left(\frac{g_1^2 g_3^2 g_5^2\kappa_a}{g_2^2 g_4^2 g_6^2} + \kappa_{b,i}\right)^2$
$N = 6$	$\left(\frac{4g_1^2 g_3^2 g_5^2 g_7^2}{g_2^2 g_4^2 g_6^2\kappa_b} + \kappa_{a,i}\right)^2$	$\left(\frac{4g_1^2 g_3^2 g_5^2 g_7^2}{g_2^2 g_4^2 g_6^2\kappa_a} + \kappa_{b,i}\right)^2$
$N = 7$	$\left(\frac{g_1^2 g_3^2 g_5^2 g_7^2\kappa_b}{g_2^2 g_4^2 g_6^2 g_8^2} + \kappa_{a,i}\right)^2$	$\left(\frac{g_1^2 g_3^2 g_5^2 g_7^2\kappa_a}{g_2^2 g_4^2 g_6^2 g_8^2} + \kappa_{b,i}\right)^2$

TABLE II. Example of explicit calculations of $\kappa_{a,\mathrm{ex}}$ and $\kappa_{b,\mathrm{ex}}$ using Eq. 18.

Given no intrinsic loss and resonant modes, the general condition for bireflectionless N-stage transduction is given in Table I. Using Eq. 18, we achieve the conditional equations for the external loss rates of the end modes, shown above in Table II. Note that the total loss rate of each mode is given by the sums $\kappa_a = \kappa_{a,\mathrm{ex}} + \kappa_{a,i}$ and $\kappa_b = \kappa_{b,\mathrm{ex}} + \kappa_{b,i}$.

IV. DISCUSSION AND FURTHER RESEARCH

Although having zero intrinsic loss and resonant modes greatly simplifies the equations, it is an ideal scenario. In reality, there may be some detuning and dissipation between internal modes. We are looking to expand our results to include conditions for the nonresonant case and cases where intrinsic loss is non-zero. One important question to consider is whether it is possible to even achieve maximum efficiency with internal loss. Although the scattering matrix formalism becomes cumbersome in multi-stage transduction, it is necessary and precise when calculating exact transmission and reflection rates between any two modes.

ACKNOWLEDGMENTS

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Finally, we would like to thank National Taiwan University, the Physics Department at NTU, and the Office of International Affairs for arranging all of the events and logistics related to this program.

Appendix A: Single Stage Transduction

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S_{11}} & \boldsymbol{S_{12}} & \boldsymbol{S_{13}} \\ \boldsymbol{S_{21}} & \boldsymbol{S_{22}} & \boldsymbol{S_{23}} \\ \boldsymbol{S_{31}} & \boldsymbol{S_{32}} & \boldsymbol{S_{33}} \end{pmatrix} \tag{A1}$$

In order to calculate the loss rates satisfying the bireflectionless condition, we need to set $R_1 = |S_{11}|^2 = R_2 = |S_{22}|^2 = R_3 = |S_{33}|^2 = 0$. Thus, we calculate the matrix elements shown in Figures 3 and 4.

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[1] H. Takesue, S. D. Dyer, M. J. Stevens, V. Verma, R. P. Mirin, and S. W. Nam, enQuantum teleportation over 100 km of fiber using highly efficient superconducting nanowire

$$S_{11}=\begin{pmatrix} 1-\frac{\kappa_{1,e}\left(g^2+\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} & -\frac{\sqrt{\kappa_{1,e}}\sqrt{\kappa_{2,e}}\left(g\Delta_3-\frac{1}{2}\mathrm{i}g\kappa_{3,e}-\frac{1}{2}\mathrm{i}g\kappa_{3,i}\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} \\ -\frac{\sqrt{\kappa_{1,e}}\sqrt{\kappa_{2,e}}\left(g\Delta_3-\frac{1}{2}\mathrm{i}g\kappa_{3,e}-\frac{1}{2}\mathrm{i}g\kappa_{3,i}\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} & 1-\frac{\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\kappa_{2,e}\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} \end{pmatrix}$$

$$S_{22}=\begin{pmatrix} 1-\frac{\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\kappa_{3,e}}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} & \frac{g^2\sqrt{\kappa_{1,i}}\sqrt{\kappa_{3,e}}}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} \\ \frac{g^2\sqrt{\kappa_{1,e}}\sqrt{\kappa_{3,i}}}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} & 1-\frac{\kappa_{1,i}\left(g^2+\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} \end{pmatrix}$$

$$S_{33}=\begin{pmatrix} 1-\frac{\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\kappa_{2,i}\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} & -\frac{\left(g\Delta_1-\frac{1}{2}\mathrm{i}g\kappa_{1,e}-\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)\sqrt{\kappa_{2,i}}\sqrt{\kappa_{3,i}}}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} \\ -\frac{\left(g\Delta_1-\frac{1}{2}\mathrm{i}g\kappa_{1,e}-\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)\sqrt{\kappa_{2,i}}\sqrt{\kappa_{3,i}}}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} & 1-\frac{\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\kappa_{3,i}}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_{1,e}+\frac{1}{2}\mathrm{i}g\kappa_{1,i}\right)+\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_{1,e}+\kappa_{1,i})\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_{2,e}+\kappa_{2,i})\right)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_{3,e}+\kappa_{3,i})\right)} \end{pmatrix}$$

FIG. 3. The diagonal matrices of the single stage transduction scattering block matrix in the most general case (considering distinct intrinsic dissipation and detuning).

$$S_{11}=\begin{pmatrix} 1-\frac{\kappa_e\left(g^2+\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} & -\frac{\kappa_e\left(g\Delta_3-\frac{1}{2}\mathrm{i}g\kappa_e-\frac{1}{2}\mathrm{i}g\kappa_i\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} \\ -\frac{\kappa_e\left(g\Delta_3-\frac{1}{2}\mathrm{i}g\kappa_e-\frac{1}{2}\mathrm{i}g\kappa_i\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} & 1-\frac{\kappa_e\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} \end{pmatrix}$$

$$S_{22}=\begin{pmatrix} 1-\frac{\kappa_e\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} & \frac{g^2\sqrt{\kappa_e}\sqrt{\kappa_i}}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} \\ \frac{g^2\sqrt{\kappa_e}\sqrt{\kappa_i}}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} & 1-\frac{\kappa_i\left(g^2+\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} \end{pmatrix}$$

$$S_{33}=\begin{pmatrix} 1-\frac{\kappa_i\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} & -\frac{\kappa_i\left(g\Delta_1-\frac{1}{2}\mathrm{i}g\kappa_e-\frac{1}{2}\mathrm{i}g\kappa_i\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} \\ -\frac{\kappa_i\left(g\Delta_1-\frac{1}{2}\mathrm{i}g\kappa_e-\frac{1}{2}\mathrm{i}g\kappa_i\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} & 1-\frac{\kappa_i\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)}{-\mathrm{i}g\left(-g\Delta_1+\frac{1}{2}\mathrm{i}g\kappa_e+\frac{1}{2}\mathrm{i}g\kappa_i\right)+\left(\mathrm{i}\Delta_3+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(g^2+\left(\mathrm{i}\Delta_1+\frac{1}{2}(\kappa_e+\kappa_i)\right)\left(\mathrm{i}\Delta_2+\frac{1}{2}(\kappa_e+\kappa_i)\right)\right)} \end{pmatrix}$$

FIG. 4. The diagonal matrices of the single stage transduction scattering block matrix considering equal internal loss across all modes.

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