Physics 452: Assignment 1 - Quantum Circuits

Lecturer: Cheng Chin

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Part 1: Quantum Composer Practice

Goal: We will use the **IBM Quantum Composer** to design, test, and validate quantum circuits and algorithms on both simulators and physical hardware.

Online Resource

- **IBM Quantum Composer:** https://quantum.ibm.com/composer
- **IBM Quantum Learning:** https://learning.quantum.ibm.com/
- **Documentation:** https://docs.quantum.ibm.com/

1. **Entanglement and Correlations (The Bell Test)**
   Construct a Bell State $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ in the Composer.
   - (a) Add a measurement gate to both qubits. Copy and paste the diagram.
   - (b) Run the experiment using the **"Inspect"** tool (Simulator). What are the probabilities of 00 to 11 outcomes?
   - (c) Now, add an X gate to q_0 *after* the entanglement but *before* the measurement. What is the final state and what happens to the correlations?

2. **Hardware Topology and Gate Decomposition**
   Open the **"Compute Resources"** tab and select a real IBM processor (e.g., *ibm_osaka*). Look at the **Coupling Map**.

   Identify two qubits that are **not** directly connected. We now attempt to place a CNOT gate between q_0 and q_3 in the Composer.
   - (a) Design the circuit to implement the CNOT gate based on operations between neighbors.
   - (b) How many gates do you need to resolve this lack of connectivity?

3. **Measuring Interference (The Mach-Zehnder Analogue)**
   Create a circuit: $H \to Z \to H$ on q_0.
   - (a) Predict the outcome before running.
   - (b) In the Composer, change the middle gate to a $P(\theta)$ gate. Use the **Probability View** to find the value of θ where the probability of measuring $|1\rangle$ is exactly 0.5.

4. **State Engineering: NOON, Cat, and W-States**
   Use the Composer to design circuits for the following 3-qubit states:

- (a) **Cat State (GHZ):** $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.
- (b) **NOON State:** $|\Psi\rangle = \frac{1}{\sqrt{2}}(|3,0\rangle + |0,3\rangle)$ (In the circuit model, this is identical to the GHZ state for 3 qubits).
- (c) **W-State:** $|\Psi\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$. *Hint: You will need to use a custom $R_y(\theta)$ gate, and determine θ.*

Part 2: (Paper and Pen) Single and Two-Qubit Operation

5. **Question:** Show that the Hadamard gate H can be expressed as a rotation on the Bloch sphere. Specifically, prove the identity $H = \frac{1}{\sqrt{2}}(X + Z)$. Then, calculate $H|0\rangle$ and find its Bloch vector $\mathbf{b} = (\langle X\rangle, \langle Y\rangle, \langle Z\rangle)$.

6. **Question:** Calculate the expectation value of the observable $X \otimes X$ for the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

7. **Question:** Starting with the product state $|00\rangle$, apply H to the first qubit and then a CNOT $(1 \to 2)$. Write out the state vector at each step and show it is a Bell state.

Part 3: (Paper and Pen) Advanced Questions

8. **Question:** Consider $|\Psi\rangle = \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle)$. If you measure the third qubit and obtain result "1", what is the normalized state of the remaining two qubits? Is it entangled?

9. **Question:** For the GHZ state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, calculate the reduced density matrix ρ_1 for the first qubit. What is the length of the resulting Bloch vector?
Solution: $\rho_1 = \mathrm{Tr}_{23}(|GHZ\rangle\langle GHZ|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}I$. The Bloch vector $\mathbf{b} = (0, 0, 0)$, so the length $|\mathbf{b}| = 0$, indicating a maximally mixed state.

10. **Question: The Quantum XY Model and Non-Commuting Evolution**
The 2D XY model is a fundamental framework in condensed matter physics used to describe phase transitions in magnetic systems and superfluidity. In its simplest 2-qubit form, with $J > 0$, the Hamiltonian represents the exchange interaction between neighboring spins:

$$H = J(X_1X_2 + Z_1Z_2) \tag{1}$$

Note that $[X_1X_2, Z_1Z_2] \neq 0$, making the evolution $U(t) = e^{-iHt/\hbar}$ non-trivial.

   - (a) Show that the Bell states $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$ are the eigenstates of this Hamiltonian. List the eigenstates in order of increasing energy (from most negative to most positive eigenvalue).
   - (b) Provide a physical intuition for why the state with the highest energy and the state with the lowest energy have the symmetry they do. Consider the alignment (parallel vs. anti-parallel) of the spins in the X and Z bases.
   - (c) If the system is initialized in the state $|\Psi(0)\rangle = |01\rangle$, calculate the state vector $|\Psi(t)\rangle$ after time t. Does the state evolve or remain stationary?
   - (d) Describe a circuit that implements this evolution *exactly* (without Trotterization) by using a basis transformation.

Answer Key

Part 1: Quantum Composer Practice

1. Entanglement and Correlations (The Bell Test)

(a) Circuit Construction: To construct $|\Phi^+\rangle$, apply a Hadamard gate H to q_0, followed by a CNOT gate with q_0 as control and q_1 as target. Add measurement gates to both qubits.

(b) Probabilities: In an ideal simulator, the outcomes $|00\rangle$ and $|11\rangle$ each have a probability of 0.5 (50%). Outcomes $|01\rangle$ and $|10\rangle$ have 0% probability.

(c) X gate addition: Adding an X gate to q_0 after entanglement transforms the state:

$$X_0|\Phi^+\rangle = X_0\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) = |\Psi^+\rangle$$

The correlations remain perfect but are now anti-correlated; measuring one qubit as 0 guarantees the other is 1.

2. Hardware Topology and Gate Decomposition

(a) Connectivity Design: If q_0 and q_3 are not connected, we use SWAP gates to move the state. A common decomposition for a linear chain $(0-1-2-3)$ is: $SWAP(q_1, q_2) \rightarrow SWAP(q_2, q_3) \rightarrow CNOT(q_0, q_3) \dots$.

(b) Gate Count: A single SWAP gate decomposes into 3 CNOT gates. To move a qubit across one gap and back requires 2 SWAPs (6 CNOTs) plus the original logical CNOT, totaling at least 7 CNOT gates.

3. Measuring Interference

(a) Prediction: $HZH|0\rangle = HZ|+\rangle = H|-\rangle = |1\rangle$. The outcome is $|1\rangle$ with 100% probability.

(b) Phase Gate: For $HP(\theta)H|0\rangle$, the probability of measuring $|1\rangle$ is $P(1) = \sin^2(\theta/2)$. Setting $P(1) = 0.5$ requires $\sin^2(\theta/2) = 0.5 \implies \theta = \pi/2$ (or 90°).

4. State Engineering

- **Cat/GHZ State:** $H(q_0) \rightarrow CNOT(q_0, q_1) \rightarrow CNOT(q_1, q_2)$.
- **W-State:** Apply $R_y(\theta)$ with $\theta = 2\cos^{-1}(1/\sqrt{3})$ to q_0, followed by a controlled-Hadamard and CNOTs to distribute the single excitation.

Part 2: Single and Two-Qubit Operation

5. Hadamard as Bloch Rotation

Identity: $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$. Since $X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, then $\frac{1}{\sqrt{2}}(X + Z) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = H$.

Bloch Vector: $H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

- $\langle X\rangle = \langle +|X|+\rangle = 1$

- $\langle Y \rangle = \langle +|Y|+ \rangle = 0$
- $\langle Z \rangle = \langle +|Z|+ \rangle = 0$

The Bloch vector is $b = (1, 0, 0)$.

6. Expectation of $X \otimes X$

For $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$:

$$(X \otimes X)|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle) = |\Phi^+\rangle$$

The expectation value is $\langle \Phi^+|X \otimes X|\Phi^+\rangle = \langle \Phi^+|\Phi^+\rangle = 1$.

Part 3: Advanced Questions

8. Measurement and Entanglement

$|\Psi\rangle = \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle)$. Measuring $q_2 = 1$ collapses the state to the terms ending in '1': $|011\rangle + |101\rangle$. Normalized state: $|\psi_{rem}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. This is the Bell state $|\Psi^+\rangle$, which is **maximally entangled**.

10. Quantum XY Model

(a) Eigenstates: The Hamiltonian is $H = J(X_1X_2 + Z_1Z_2)$.

- $H|\Phi^+\rangle = J(1 + 1)|\Phi^+\rangle = 2J|\Phi^+\rangle$ (Energy $2J$)
- $H|\Phi^-\rangle = J(-1 + 1)|\Phi^-\rangle = 0$ (Energy 0)
- $H|\Psi^+\rangle = J(1 - 1)|\Psi^+\rangle = 0$ (Energy 0)
- $H|\Psi^-\rangle = J(-1 - 1)|\Psi^-\rangle = -2J|\Psi^-\rangle$ (Energy $-2J$)

Order: $|\Psi^-\rangle, \{|\Phi^-\rangle, |\Psi^+\rangle\}, |\Phi^+\rangle$.

(b) Intuition: The lowest energy state $|\Psi^-\rangle$ has anti-parallel spins in both X and Z bases, minimizing the $J(X_1X_2 + Z_1Z_2)$ energy for $J > 0$.

(c) Evolution of $|01\rangle$: $|01\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle)$.

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(e^{-i(0)t/\hbar}|\Psi^+\rangle + e^{-i(-2J)t/\hbar}|\Psi^-\rangle) = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + e^{i2Jt/\hbar}|\Psi^-\rangle)$$

The state **evolves** as it is not a pure eigenstate.