Physics 452: Quantum Optics and Quantum Gases

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Week 1: Quantum Circuits and Single Qubit Foundations

1 Introduction

Why do we study quantum many-body physics?

- **More is different** (P.W. Anderson, 1972): Emergent phenomena in large systems cannot be understood by solely looking at individual components.
- **The Qubit as a Resource:** Serves as the fundamental unit for **quantum computation**, **quantum networking**, **quantum metrology**, and **quantum simulation**.
- **Many-body systems:** We explore statistical phases such as **superconductivity**, the emergence of **quasi-particles**, and **quantum hydrodynamics**.

This course integrates quantum optics, quantum many-body physics, and atom scattering through lectures, discussions and computational projects. The course will offer fundamental understandings and collaborative simulation of essential many-body platforms: Project 1: Quantum Circuit Project 2: Hubbard Lattice Model Project 3: Multichannel Scattering

2 Assignment and AI-assisted Project

This course bridges conventional physics with modern computational research:

- **Assignment**: We will have standard "paper and pen" assignments which serve to anchor the abstract computational work in rigorous analytical derivation.
- **AI-Assisted Project:** We encourage the use of AI to perform research projects and optimize code.
- **UChicago Quantum Toolbox:** We will demonstrate how to use **Google Colab**, **GitHub**, and **Streamlit** to generate quantum apps. Outstanding applications will be curated for the departmental quantum toolbox. See Supplement

Topic 1: Quantum Circuits, Algorithms & Error Correction

Goals:

- Master the circuit model and the transition from ideal algorithms to NISQ-era challenges like error correction and mitigation.
- Learn and master the **IBM Quantum Composer** to design, test, and validate quantum circuits and algorithms on both simulators and physical quantum hardware.

3 Class 1: Hilbert Space and the Single Qubit

3.1 The Quantum State and Hilbert Space ($\mathcal{H}$)

As John Preskill states: *"A state is a complete description of a physical system. In quantum mechanics, a state is a ray in a Hilbert space."* For a single qubit, the state is a normalized vector in $\mathbb{C}^2$:

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle \tag{1}$$

The global phase is physically irrelevant; only the relative phase $e^{i\phi}$ and the probability amplitudes matter.

3.2 The Bloch Vector and Density Matrix

The state of a qubit can be represented as a vector $\mathbf{b}$ in 3D Euclidean space:

$$\mathbf{b} = \langle\vec{\sigma}\rangle = (\langle X\rangle, \langle Y\rangle, \langle Z\rangle) \tag{2}$$

- **Pure States:** $|\mathbf{b}| = 1$. The state is a point on the surface of the **Bloch Sphere**.
- **Mixed States:** $|\mathbf{b}| < 1$. This represents statistical uncertainty (decoherence).
- **Density Matrix:** The operator $\rho = |\psi\rangle\langle\psi|$ (for pure states) or $\rho = \frac{1}{2}(I + \mathbf{b}\cdot\vec{\sigma})$ generalizes the state description.

3.3 Dynamics: From Hamiltonians to Unitary Gates

The energy of the system is described by the **Hamiltonian** (H). Dynamics are governed by the Schrödinger equation, resulting in **Unitary operations** (U):

$$U(t) = e^{-iHt/\hbar}, \quad UU^\dagger = I \tag{3}$$

Geometrically, U represents a **rotation** on the Bloch sphere. For a Hamiltonian $H = \frac{\hbar\Omega}{2}\hat{n}\cdot\vec{\sigma}$, the state rotates around axis $\hat{n}$ by angle $\theta = \Omega t$.

3.4 Fundamental Single-Qubit Gates

Table 1: Essential Single-Qubit Gates and Bloch Vector Transformations

Symbol	**Name**	**Generator** (H)	**Matrix Form**	**Bloch Map** $(b_x, b_y, b_z) \to$
X	NOT	$\frac{\pi}{2}\sigma_x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$(b_x, -b_y, -b_z)$
Y	Pauli-Y	$\frac{\pi}{2}\sigma_y$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$(-b_x, b_y, -b_z)$
Z	Phase flip	$\frac{\pi}{2}\sigma_z$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$(-b_x, -b_y, b_z)$
H	Hadamard	$\frac{\pi}{2}\frac{\sigma_x+\sigma_z}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$(b_z, -b_y, b_x)$
S	$\pi/4$ gate	$\frac{\pi}{4}\sigma_z$	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	$(-b_y, b_x, b_z)$
T	$\pi/8$ gate	$\frac{\pi}{8}\sigma_z$	$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$	$(\frac{b_x-b_y}{\sqrt{2}}, \frac{b_x+b_y}{\sqrt{2}}, b_z)$

4 Class 2: Multi-Qubit Systems and Entanglement

4.1 Composite Systems: The Tensor Product ($\otimes$)

When we combine n qubits, the Hilbert space dimensions grow exponentially as 2^n. The state space is a **Tensor Product** $\mathcal{H}_{total} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$. For two qubits $|\psi\rangle$ and $|\phi\rangle$:

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix} \tag{4}$$

The computational basis is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. A general state is:

$$|\Psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \tag{5}$$

4.2 Separability vs. Entanglement

- **Separable States:** Can be written as a product $|\psi\rangle_A \otimes |\phi\rangle_B$. Knowledge of the whole is exactly the sum of its parts.
- **Entangled States:** Cannot be factored. These represent non-local correlations that exceed classical limits (The EPR Paradox). The most famous are the **Bell States**:

$$|\Phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \quad |\Psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \tag{6}$$

4.3 Multi-Qubit Gate Operations

To create entanglement, we require interactions.

- **CNOT (Controlled-NOT):** Flips the target qubit if the control is $|1\rangle$. It is its own inverse (**Self-inversion**).

$$CNOT = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{7}$$

- **CZ (Controlled-Z):** Adds a π phase if both qubits are $|1\rangle$. This gate is symmetric between control and target.

- **SWAP Gate:** Physically exchanges the states of two qubits.

$$SWAP(q_i, q_j) = CNOT(i, j) \cdot CNOT(j, i) \cdot CNOT(i, j) \tag{8}$$

4.4 Universal Gate Sets

A set of gates is **Universal** if any unitary operation can be approximated to arbitrary precision.

- **Standard Set:** $\{H, S, T, CNOT\}$.

- *Note:* H and T are required for single-qubit universality; $CNOT$ provides the necessary multi-qubit interaction.

4.5 Quantum Circuit Equivalence and Basic Identities

Understanding how gates transform is vital for circuit optimization and hardware transpilation:

- **Basis Rotations:** $HXH = Z$ and $HZH = X$.

- **CNOT Reverse:** Applying H gates to all ports of a $CNOT$ flips the direction of the control and target.

4.6 Phase Kickback

Phase kickback occurs when a controlled operation is applied to a target qubit in an eigenstate of that operation. The eigenvalue (phase) is "kicked back" to the control qubit. *Example:* If the target of a $CNOT$ is in the $|-\rangle$ state, the X operation yields a -1 phase, which effectively applies a Z gate to the control qubit.

4.7 Measurement, Partial Trace, and Fidelity

- **Partial Measurement:** Measuring only one qubit of an entangled pair causes **Partial Collapse**. For $|\Phi^+\rangle$, measuring q_1 as "0" forces q_2 into $|0\rangle$ instantaneously. This is the basis for **Quantum Teleportation**.

- **Partial Trace:** Used to find the state of a subsystem ($\rho_A = \text{Tr}_B(\rho_{AB})$). For a maximally entangled Bell state, the partial trace yields a **Maximally Mixed State** ($\rho_A = I/2$).

- **Measurement Fidelity:** In real QPUs, measurement is imperfect. Fidelity $\mathcal{F}$ quantifies how close the measured result is to the ideal state.

Table 2: Two-Qubit Gates

Gate	Generator (H)	Matrix	Logic
$CNOT$	$\frac{\pi}{4}(I-\sigma_{z1})(I-\sigma_{x2})$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$	Flip q_2 if $q_1=\|1\rangle$
CZ	$\frac{\pi}{4}(I-\sigma_{z1})(I-\sigma_{z2})$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$	Phase π if $\|11\rangle$
$SWAP$	$\frac{\pi}{4}(\vec{\sigma}_1\cdot\vec{\sigma}_2)$	$\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$	$\|ab\rangle\to\|ba\rangle$
$iSWAP$	$\frac{\pi}{4}(\sigma_{x1}\sigma_{x2}+\sigma_{y1}\sigma_{y2})$	$\begin{pmatrix}1&0&0&0\\0&0&i&0\\0&i&0&0\\0&0&0&1\end{pmatrix}$	Swap with phase i

5 Class 3: Quantum algorithm

5.1 Introduction: The Nature of Quantum Speedup

Quantum algorithms do not run faster, they utilize collective interference to cancel out incorrect computational paths and amplify the correct one. We categorize quantum advantage into three primary domains:

- **Communication:** Moving information using entanglement as a resource.
- **Simulation:** Mapping physical Hamiltonians directly to qubit registers.
- **Computation:** Using structural properties (like periodicity) to achieve exponential speedups.

Algorithm	Qubits	Classical	Quantum	Speedup
Teleportation	3	N/A	$O(1)$	Logic
Deutsch-Jozsa	$n+1$	$O(2^{n-1})$	1 Query	Exp.
Grover Search	$n+1$	$O(N)$	$O(\sqrt{N})$	Quad.
QFT	n	$O(n2^n)$	$O(n^2)$	Exp.
Fermi-Hubbard	$2\times$ Sites	$O(e^n)$	Poly(n)	Exp.
Shor's (Factoring)	$3n+1$	$O(e^{n^{1/3}})$	$O(n^3)$	Exp.

—

5.2 Quantum Communication: Teleportation

Quantum Teleportation is the "disembodied" transfer of an unknown state $|\psi\rangle$ from Alice to Bob. It is the fundamental protocol for the **Quantum Internet**.

1. **Resource:** Alice and Bob share a Bell pair $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

2. **Alice's Operation:** Alice performs a **Bell State Measurement (BSM)** on her qubit and the unknown state $|\psi\rangle$. This involves a CNOT, a Hadamard, and measurement in the Z-basis.

3. **Classical Feed-forward:** Alice sends two classical bits (m_1, m_2) to Bob.

4. **Bob's Correction:** Depending on the bits, Bob applies $X^{m_2}Z^{m_1}$ to his half of the Bell pair.

$$\text{Result: } |\psi\rangle_{Alice} \to |\psi\rangle_{Bob} \tag{9}$$

Quantum teleportation allows the transfer of an unknown qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ from Alice to Bob using a shared Bell pair and classical communication.

Step 1: Initial State

Alice possesses qubit 1 ($|\psi\rangle$) and qubit 2 (half of the Bell pair). Bob possesses qubit 3 (the other half). The initial state of the 3-qubit system is:

$$\begin{aligned}
|\Psi_0\rangle &= |\psi\rangle_1 \otimes |\Phi^+\rangle_{23} \\
&= (\alpha|0\rangle_1 + \beta|1\rangle_1) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{23} + |11\rangle_{23}) \\
&= \frac{1}{\sqrt{2}}[\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle]
\end{aligned}$$

Step 2: Alice's Bell Measurement Operations

Alice applies a $CNOT_{12}$ followed by a Hadamard H_1 to her qubits.

1. **After $CNOT_{12}$:** $|\Psi_1\rangle = \frac{1}{\sqrt{2}}[\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle]$

2. **After H_1:** Substitute $|0\rangle \to \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|1\rangle \to \frac{|0\rangle-|1\rangle}{\sqrt{2}}$:

$$\begin{aligned}
|\Psi_2\rangle = \frac{1}{2}[&\alpha(|0\rangle + |1\rangle)|00\rangle + \alpha(|0\rangle + |1\rangle)|11\rangle \\
&+\beta(|0\rangle - |1\rangle)|10\rangle + \beta(|0\rangle - |1\rangle)|01\rangle]
\end{aligned}$$

Step 3: State Reorganization and Measurement

Regrouping the state by Alice's possible measurement outcomes on qubits 1 and 2:

$$\begin{aligned}
|\Psi_2\rangle = \frac{1}{2}\Big[&|00\rangle_A(\alpha|0\rangle + \beta|1\rangle)_B + |01\rangle_A(\alpha|1\rangle + \beta|0\rangle)_B \\
&+|10\rangle_A(\alpha|0\rangle - \beta|1\rangle)_B + |11\rangle_A(\alpha|1\rangle - \beta|0\rangle)_B\Big]
\end{aligned}$$

When Alice measures her qubits, Bob's qubit is projected into a state that is unitary-equivalent to $|\psi\rangle$. Alice sends bits m_1m_2 and Bob applies $Z^{m_1}X^{m_2}$ to recover the original state.

No-Communication Theorem

Classical communication is necessary because a local operation performed by Alice on her subsystem A cannot be detected by Bob through any local measurement on his subsystem B.

Consider a bipartite state ρ_{AB} in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Bob's local physical reality is entirely described by his **reduced density matrix**:

$$\rho_B = \text{Tr}_A(\rho_{AB}) \tag{10}$$

Any local observable M_B measured by Bob has an expectation value:

$$\langle M_B \rangle = \text{Tr}(M_B \rho_B) = \text{Tr}((I_A \otimes M_B)\rho_{AB}) \tag{11}$$

Suppose Alice applies a local unitary U_A to her qubits. The new global state is $\rho'_{AB} = (U_A \otimes I_B)\rho_{AB}(U_A^\dagger \otimes I_B)$. Bob's new reduced density matrix ρ'_B is:

$$\rho'_B = \text{Tr}_A \left((U_A \otimes I_B)\rho_{AB}(U_A^\dagger \otimes I_B) \right)$$

Using the cyclic property of the partial trace, $\text{Tr}_A((U_A \otimes I)\rho) = \text{Tr}_A(\rho(U_A \otimes I))$, we get:

$$\rho'_B = \text{Tr}_A \left((U_A^\dagger U_A \otimes I_B)\rho_{AB} \right)$$
$$\rho'_B = \text{Tr}_A(I_A \otimes I_B \rho_{AB}) = \rho_B$$

Since $\rho'_B = \rho_B$, Bob's measurement statistics remain unchanged. —

5.3 Quantum Simulation: The Fermi-Hubbard Model

As experimentalists, we use qubits to simulate condensed matter systems that are numerically "intractable" due to the exponential growth of the Hilbert space.

In a 2D lattice, we map the creation/annihilation operators of fermions to qubit operations using the **Jordan-Wigner transformation**. The Hamiltonian for a single site interaction is:

$$H = -t \sum_{\langle i,j \rangle, \sigma} (c_{i\sigma}^\dagger c_{j\sigma} + \text{h.c.}) + U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{12}$$

Trotterization: Because the kinetic (t) and potential (U) terms do not commute, we simulate time evolution $U(t) = e^{-iHt}$ by breaking it into small steps:

$$e^{-i(A+B)\Delta t} \approx e^{-iA\Delta t} e^{-iB\Delta t} + \mathcal{O}(\Delta t^2) \tag{13}$$

Each step corresponds to a sequence of CZ, $SWAP$, and rotation gates on our 10-qubit simulator.

6 The Jordan-Wigner Transformation for Hubbard Models

To simulate fermions on a quantum computer, we must map the anti-commuting fermionic operators $\{c_i, c_j^\dagger\} = \delta_{ij}$ to the commuting algebra of qubits $[X_i, Y_j] = 0$. The **Jordan-Wigner (JW) Transformation** provides this dictionary by encoding the fermionic phase information into "Z-strings."

6.1 Mapping of Fock Space to Qubit Space

We map each fermionic mode (defined by site i and spin $\sigma \in \{\uparrow, \downarrow\}$) to a single qubit. For a system of L sites, we require $2L$ qubits. The occupation of a mode is mapped to the computational basis:

- **Empty state:** $|n_{i\sigma} = 0\rangle \rightarrow |0\rangle_q$
- **Occupied state:** $|n_{i\sigma} = 1\rangle \rightarrow |1\rangle_q$

The creation operator $c_j^\dagger$ at index j is represented as:

$$c_j^\dagger \rightarrow \left(\bigotimes_{k=1}^{j-1} Z_k\right) \otimes \sigma_j^+, \quad \sigma^+ = \frac{1}{2}(X - iY) \tag{14}$$

The "string" of Z operators ensures that the operation picks up a (-1) phase for every fermion located at an index $k < j$, preserving the global anti-symmetry requirement.

6.2 Mapping Hamiltonian Terms to Gates

The Fermi-Hubbard Hamiltonian $H = H_{hop} + H_{int}$ is decomposed as follows:

6.2.1 Tunneling (Hopping) Terms

The hopping between adjacent sites i and j in a single spin sector is given by $H_{hop} = -t(c_i^\dagger c_j + c_j^\dagger c_i)$. Under JW mapping (assuming $j = i + 1$):

$$-t(c_i^\dagger c_{i+1} + \text{h.c.}) \rightarrow -\frac{t}{2}(X_i X_{i+1} + Y_i Y_{i+1}) \tag{15}$$

The unitary evolution $e^{-iH_{hop}\Delta t}$ is implemented using an **XY-gate** or a combination of CNOTs and R_z rotations. In many architectures, this is natively supported as the **iSWAP** or **fSim** gate.

6.2.2 Interaction Terms and Phase Gates

The interaction $Un_{i\uparrow}n_{i\downarrow}$ evolution is given by $e^{-i\frac{U\Delta t}{4}Z_i Z_j}$ (neglecting local R_z corrections). This is a **Controlled-Phase** operation. While a standard CZ gate applies a fixed π phase, the Hubbard model requires a parameterized rotation:

$$U_{int}(\Delta t) = \exp\left(-i\frac{U\Delta t}{4} Z_1 Z_2\right) \tag{16}$$

On hardware, this is typically synthesized via a CNOT-sandwich or a parameterized $fSim$ gate.

6.3 Example: Trotterized Evolution Unit Step

For a 2-site Hubbard model (4 qubits), a single first-order Trotter step $U(\Delta t) \approx e^{-iH_{int}\Delta t}e^{-iH_{hop}\Delta t}$ is executed in the following sequence:

1. **Hopping Layer:** Apply $e^{i\frac{t\Delta t}{2}(X_0 X_2 + Y_0 Y_2)}$ (Spin $\uparrow$) and $e^{i\frac{t\Delta t}{2}(X_1 X_3 + Y_1 Y_3)}$ (Spin $\downarrow$).
2. **Interaction Layer:** Apply $e^{-i\frac{U\Delta t}{4}Z_0 Z_1}$ (Site 1) and $e^{-i\frac{U\Delta t}{4}Z_2 Z_3}$ (Site 2).
3. **Potential Layer (Optional):** Apply R_z rotations to simulate chemical potential $\mu \sum n_i$.

7 Exercise: Trotter Error Analysis

Problem: Consider a 2-site Fermi-Hubbard model with Hamiltonian $H = T + V$, where $T = -t(c_1^\dagger c_2 + c_2^\dagger c_1)$ and $V = U(n_1 + n_2)$.

1. Use the Baker-Campbell-Hausdorff (BCH) formula to find the leading order error term ϵ for a single Trotter step $e^{-iT\Delta t}e^{-iV\Delta t}$.
2. Given your answer in (1), how does the total error scale with the number of sites N in a 1D chain?
3. **Discussion:** On your 10-qubit simulator, if the gate error is 0.1%, at what number of Trotter steps does the hardware noise overtake the theoretical Trotter accuracy?

8 Class 5: Quantum Error Correction

As in 2026, practical quantum computation on all platforms suffer from errors of various sources

- **Superconducting Qubits (Transmons):**
  - T_1 **(Relaxation):** Spontaneous decay from $|1\rangle \to |0\rangle$ due to dielectric loss.
  - T_2 **(Dephasing):** Phase drift caused by magnetic noise.
- **Trapped Ions:**
  - **SPAM Errors:** Preparation and Measurement errors during laser cooling or detection.
  - **Gate Noise:** Intensity fluctuations in lasers during Mølmer-Sørensen operations.
- **Neutral Atoms (Rydberg):**
  - **Rydberg Decay:** The finite lifetime of the highly excited Rydberg state.
  - **Atom Loss (Erasure):** Physical loss of an atom from the optical tweezer due to collisions. This is a *leakage* error where the qubit Hilbert space literally disappears.

8.1 Classical Error Correction

Classical error correction relies on the **Repetition Code**. By encoding $0 \to 000$ and $1 \to 111$, we use the principle of majority voting. If the probability of a single bit-flip is $\epsilon \ll 1$, the probability of the code failing (2 or more flips) is $3\epsilon^2$. For small ϵ, the logical error rate is significantly lower than the physical one.

8.2 The Quantum "No-Go" Constraint

Quantum Error Correction (QEC) must bypass three physical constraints:

1. **No-Cloning:** We cannot copy $|\psi\rangle$ to create redundancy. Is this always true?
2. **Measurement Collapse:** Observing the data qubits directly destroys the quantum information.
3. **Continuous Errors:** Phase drifts and small rotations are analog, whereas classical flips are digital.

9 Bit-Flip Code

Instead of cloning, we use **Entanglement** to spread information across a non-local Hilbert space (e.g., the GHZ-basis $\alpha|000\rangle + \beta|111\rangle$). By measuring **Stabilizers** (multi-qubit parity operators), we project continuous noise into discrete Pauli errors. This "digitizes" the noise, allowing us to identify and undo the error without ever learning the values of α or β. The GHZ state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ we studied previously is the physical basis for the 3-qubit repetition code.

We map a logical qubit $|\psi\rangle_L = \alpha|0\rangle + \beta|1\rangle$ into a 3-qubit physical state:

$$|\psi\rangle_L \xrightarrow{\text{Encode}} \alpha|000\rangle + \beta|111\rangle$$

. This is implemented by 1. Start with q_0 in state $|\psi\rangle$ and q_1, q_2 in $|0\rangle$. 2. Apply CNOT ($q_0 \to q_1$) and CNOT ($q_0 \to q_2$).

We detect errors without measuring the data qubits directly (which would collapse the superposition). We measure the **parity** using ancilla qubits:

- If $Z_0Z_1 = +1$ and $Z_1Z_2 = +1$, no bit-flip occurred.
- If $Z_0Z_1 = -1$ and $Z_1Z_2 = +1$, we know q_0 flipped (X_0).

10 The Surface Code

The surface code is a 2D checkerboard lattice of qubits that provides the most realistic path to fault-tolerance due to its local connectivity. In this specific implementation of the surface code, the physical qubits are mapped onto a 2D grid where the topology determines the error-correction properties:

- **Data Qubits (D):** These occupy the **centers of the faces** (plaquettes) of the square lattice. They store the actual logical quantum information.
- **Z-Measure Qubits (M_z):** These are located at the **vertices** of the grid. Each M_z ancilla is coupled to the four surrounding data qubits to perform a parity check.
- **X-Measure Qubits (M_x):** These are co-located or interleaved at the **edges/faces** to perform X-basis parity checks across the data qubit boundaries.

The fundamental "engine" of the surface code is the anti-commutation of Pauli operators: $\{X, Z\} = 0$, or $XZ = -ZX$. We define our code space as the $+1$ eigenstate of all stabilizer operators S. When an error E occurs, it is detected if and only if it anti-commutes with a stabilizer: $S(E|\psi\rangle_L) = -E(S|\psi\rangle_L) = -1(E|\psi\rangle_L)$.

To extract error information without destroying the logical state, we use ancilla qubits to perform parity checks in a cyclic process.

10.1 Gate-Level Implementation

For a stabilizer S involving four data qubits $D_1 \dots D_4$ and one ancilla A:

1. **Z-Stabilizer (detects X errors):**

$$|0\rangle_A \xrightarrow{CNOT(D_1,A)} \dots \xrightarrow{CNOT(D_4,A)} \text{Measure } Z_A$$

   The ancilla flips to $|1\rangle$ if an odd number of data qubits are in the $|1\rangle$ state.

2. **X-Stabilizer (detects Z errors):**

$$|0\rangle_A \xrightarrow{H_A} \xrightarrow{CNOT(A,D_1)} \dots \xrightarrow{CNOT(A,D_4)} \xrightarrow{H_A} \text{Measure } Z_A$$

   By surrounding the CNOTs with Hadamards, the ancilla effectively measures the parity of the data qubits in the X-basis.

In the 2D lattice, these gates must be scheduled carefully to avoid "cross-talk" or overlapping operations. A typical cycle involves a "Star" pattern of CNOTs where the ancilla interacts with its four neighbors in a specific clockwise or counter-clockwise order.

10.2 Detecting errors

1. **Z-Checks (Vertex Operators $S_v = Z_1Z_2Z_3Z_4$):** The ancilla at the vertex measures the collective Z-parity of the four adjacent data qubits.
   - If a **bit-flip** (X) occurs on data qubit i, then $S_vX_i = -X_iS_v$.
   - The vertex measurement flips from $+1$ to -1, signaling the location of the error.
   - **Conclusion:** Vertex Z-checks are the sensors for X-type errors.
2. **X-Checks (Face Operators $S_f = X_1X_2X_3X_4$):** The ancilla associated with the face measures the X-parity of the data qubits.
   - If a **phase-flip** (Z) occurs on data qubit i, then $S_fZ_i = -Z_iS_f$.
   - The measurement result flips to -1, signaling a phase error.
   - **Conclusion:** Face X-checks are the sensors for Z-type errors.

Table 3: Syndrome Detection Logic

Error Type	**Pauli Operator**	**Detected by...**	**Lattice Location**
Bit Flip	X	Z-Check (Parity)	**Vertex**
Phase Flip	Z	X-Check (Parity)	**Face/Plaquette**
Both	$Y = iXZ$	Both X and Z Checks	Both Vertex & Face

10.3 Correcting Erasure (Atom Loss)

In Rydberg systems, if an atom is lost, we detect the "hole" in the lattice. Because the *location* of the error is known, we can re-initialize a new atom and use the surrounding X and Z stabilizers to project the logical information back onto the new physical qubit. This makes the surface code particularly robust for neutral atom arrays.

10.4 Error Detection Summary

Error Type	**Pauli**	**Detected by...**	**Mathematical Logic**
Bit Flip	X	Plaquette (Z) Check	$\{X, Z\} = 0$
Phase Flip	Z	Star (X) Check	$\{Z, X\} = 0$
Both	$Y = iXZ$	Both Checks	Anti-commutes with both