Topological matter explored with quantum gases

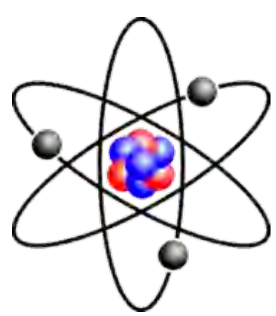
Jean Dalibard
Laboratoire Kastler Brossel
Collège de France

Geometry or topology?

Geometrical order : appears for systems with a given symmetry

Example: rotation invariance

⟶ Classification of the energy states according to their angular momentum



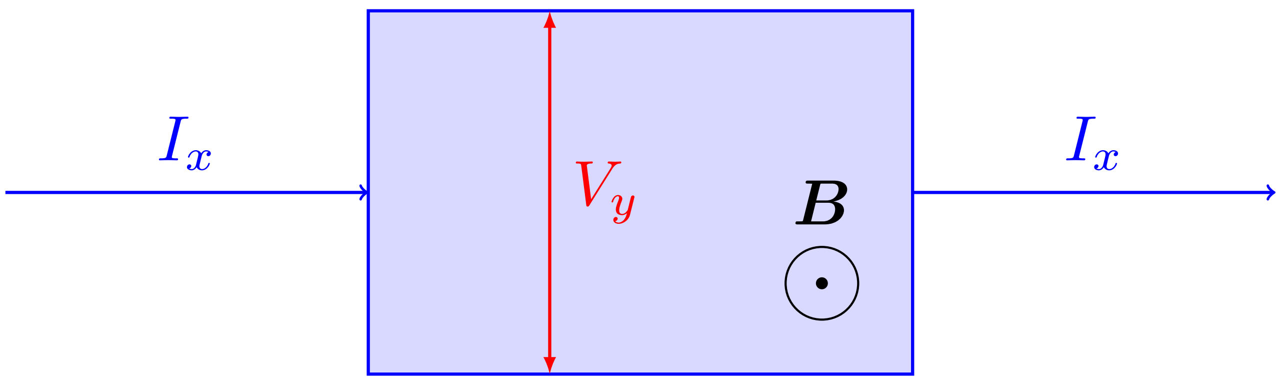
non robust:
The order is broken
by a small perturbation

Topological order: defined by classes of objects that are robust under a deformation

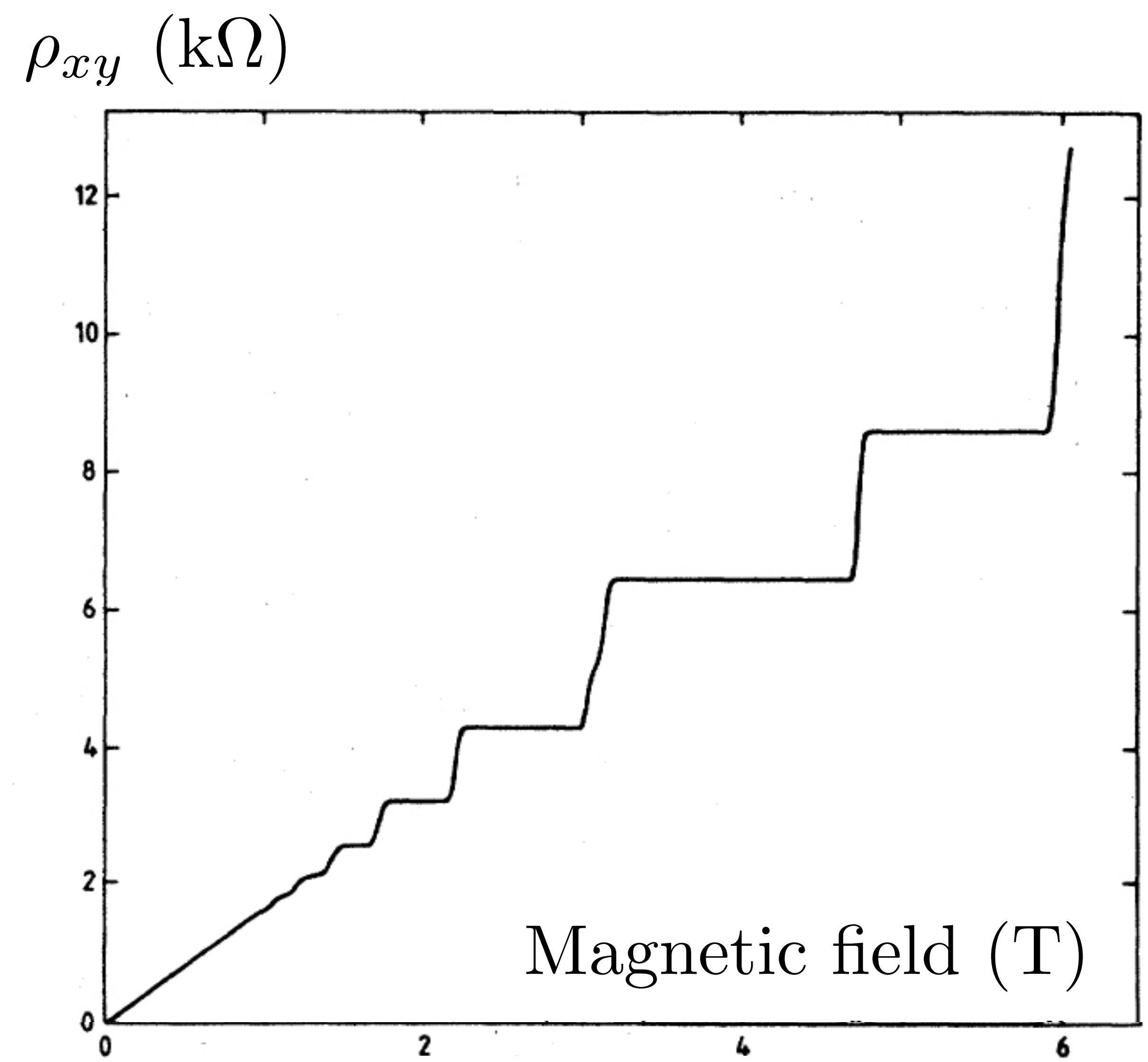
Robust by essence

A historical example

Integer Quantum Hall effect



Plateaus corresponding to a quantification of the Hall resistivity ρ_{xy}



Robust: Plateaus are still present (at the same value) in the presence of disorder

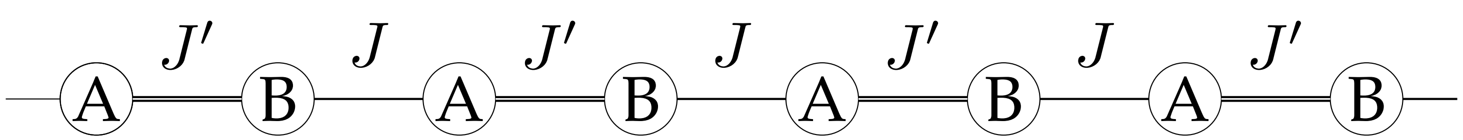
Obvious metrological interest

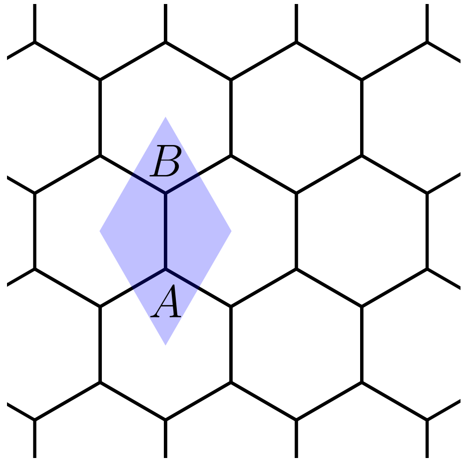
Goal of these lectures

Introduction to topological states of matter
with illustration from atom or photon gases

Simple systems:

- One or two dimensions
- Periodic systems





Optical lattices, regular arrangements of wave guides

- No interaction between particles

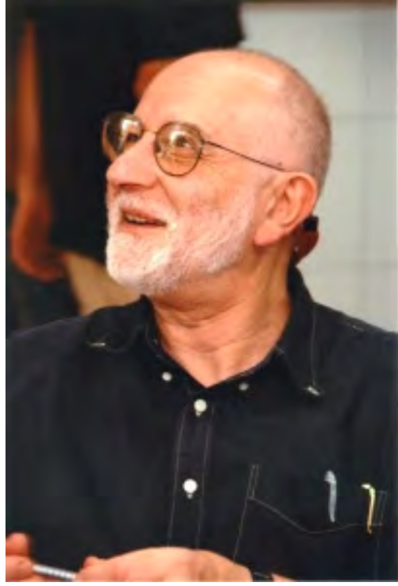
Crucial role of edge states and topological numbers

Ozawa, Price, Amo, et al., *Topological Photonics*, arXiv :1802.04173
Cooper, Dalibard, Spielman, *Topological Bands for Ultracold Atoms*, arXiv:1803.00249

1.

Berry phase in quantum mechanics

Adiabatic evolution in Quantum Physics



Berry 1984

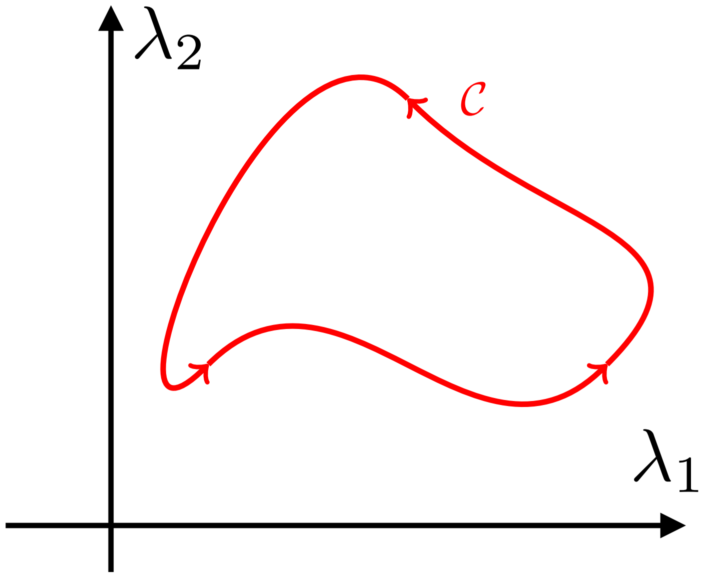
Pancharatnam (1956)
Mead & Truhlar (1979)

Quantum system (atom, molecule, macroscopic body) depending on an external parameter $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$

Evolution described by the Hamiltonian $\hat{H}_{\boldsymbol{\lambda}}$

$$\mathrm{i}\hbar \frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = \hat{H}_{\boldsymbol{\lambda}} \quad |\psi(t)\rangle$$

System prepared in an energy eigenstate, e.g. its ground state

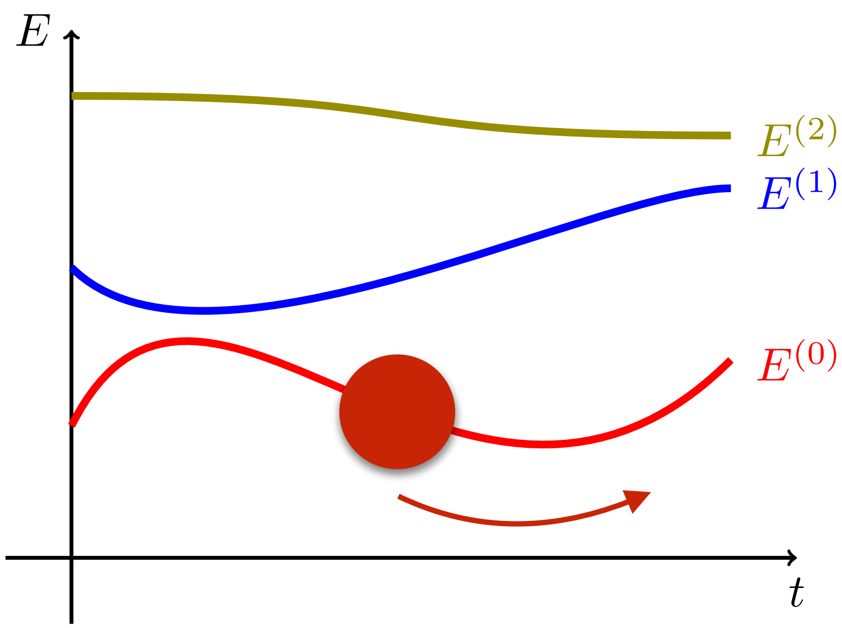


What happens when $\boldsymbol{\lambda}$ varies slowly in time and follows a closed trajectory?

$$|\psi(T)\rangle \;\approx\; \mathrm{e}^{\mathrm{i}\Phi}\; |\psi(0)\rangle$$

$$\Phi = \Phi_{\mathrm{dyn}} + \Phi_{\mathrm{geom}} \quad \begin{cases} \Phi_{\mathrm{dyn}} \propto \int E(t)\,\mathrm{d}t & \text{usual dynamical phase} \\ \Phi_{\mathrm{geom}} = ? \end{cases}$$

Geometrical phase and Berry connection



At each time *t*, we introduce a basis set of the Hamiltonian $\hat{H}_{\boldsymbol{\lambda}}$

$$|\psi_{\boldsymbol{\lambda}}^{(n)}\rangle$$

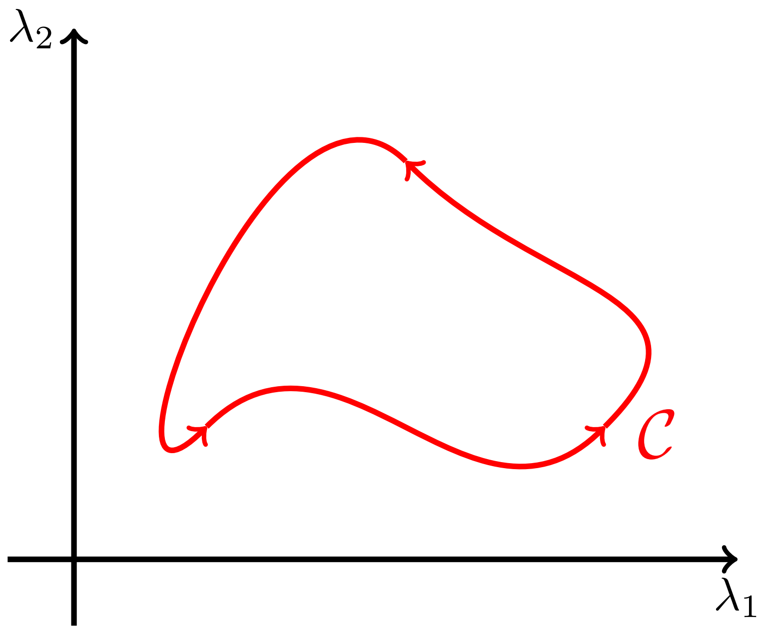
Berry connection

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(n)} = \mathrm{i}\,\langle\psi_{\boldsymbol{\lambda}}^{(n)}|\boldsymbol{\nabla}\psi_{\boldsymbol{\lambda}}^{(n)}\rangle \qquad |\boldsymbol{\nabla}\psi\rangle = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\lambda_1}|\psi\rangle \\ \frac{\mathrm{d}}{\mathrm{d}\lambda_2}|\psi\rangle \\ \vdots \end{pmatrix}$$

real since $\langle\psi|\boldsymbol{\nabla}\psi\rangle$ is purely imaginary or zero

Then : $\Phi_{\mathrm{geom}} = \int_{\boldsymbol{\lambda}(0)}^{\boldsymbol{\lambda}(t)} \boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(0)} \cdot \mathrm{d}\boldsymbol{\lambda}$

Depends on the path followed from **λ**(0) to **λ**(t),
but not on the time spent on this path.



For a closed path:

$\boldsymbol{\lambda}$: parameter in 1, 2 or 3 dimensions

$$\Phi_{\text{geom}} = \oint_{\mathcal{C}} \boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(0)} \cdot \mathrm{d}\boldsymbol{\lambda}$$

Use of Stokes formula:

$$\underbrace{\boldsymbol{\Omega}_{\boldsymbol{\lambda}}^{(n)}}_{\text{Berry curvature}} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(n)} = \mathrm{i} \, \langle \boldsymbol{\nabla} \psi_{\boldsymbol{\lambda}}^{(n)} | \times | \boldsymbol{\nabla} \psi_{\boldsymbol{\lambda}}^{(n)} \rangle$$

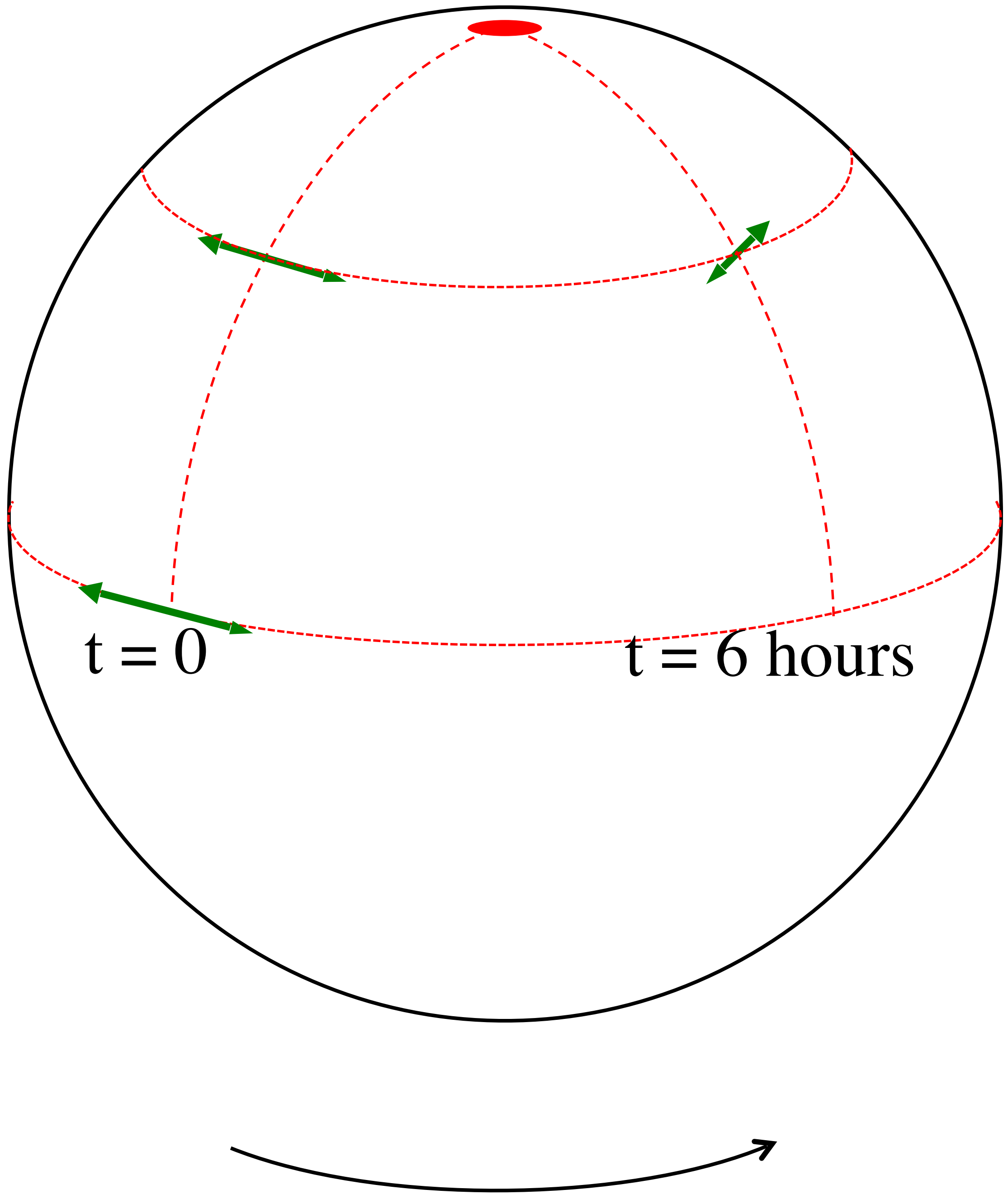
$$\longrightarrow \quad \Phi_{\text{geom}} = \iint_{\Sigma} \boldsymbol{n} \cdot \boldsymbol{\Omega}_{\boldsymbol{\lambda}}^{(n)} \, \mathrm{d}^2\lambda$$

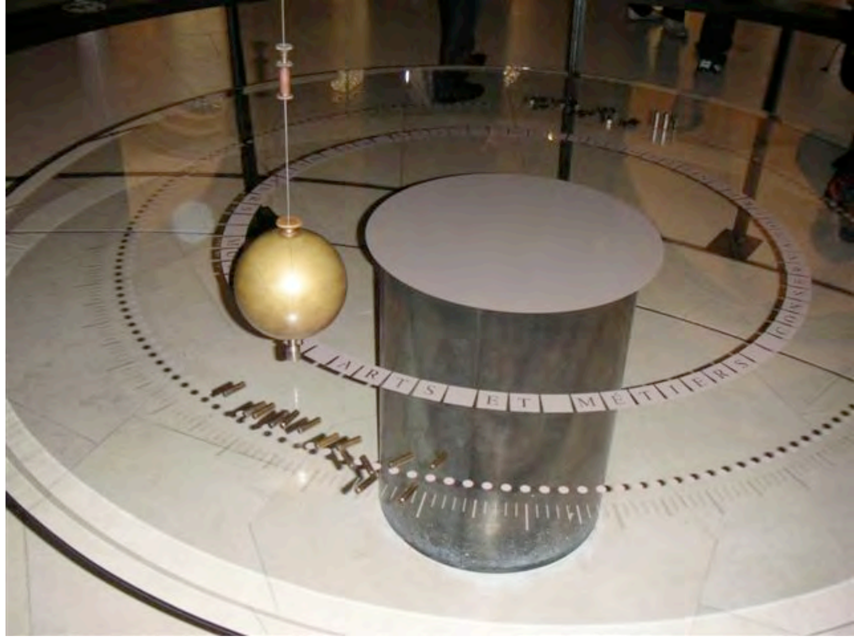
Analogy with magnetostatics:

Berry connection	$\boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}} \longleftrightarrow \boldsymbol{A}(\boldsymbol{r})$	potentiel vecteur
Berry curvature	$\boldsymbol{\Omega}_{\boldsymbol{\lambda}} \longleftrightarrow \boldsymbol{B}(\boldsymbol{r})$	magnetic field

Does not seem very interesting in 1D: mere round trip?

Foucault pendulum





Wikipedia

In 24 hours, the external parameter — the hanging point of the pendulum — makes a closed loop

The two eigenmodes for circular oscillation (clockwise and counterclockwise rotation) accumulate different phases

Correspond to a rotation of the oscillatory plane for a linear oscillation

Berry curvature and gauge invariance

Gauge change in Quantum Mechanics

$$|\psi_{\boldsymbol{\lambda}}^{(n)}\rangle \qquad \longrightarrow \qquad |\tilde{\psi}_{\boldsymbol{\lambda}}^{(n)}\rangle = \mathrm{e}^{\mathrm{i}\chi_{\boldsymbol{\lambda}}^{(n)}}\,|\psi_{\boldsymbol{\lambda}}^{(n)}\rangle$$

Berry connexion $\boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(n)} = \mathrm{i}\,\langle\psi_{\boldsymbol{\lambda}}^{(n)}|\boldsymbol{\nabla}\psi_{\boldsymbol{\lambda}}^{(n)}\rangle$ is not invariant in a gauge change:

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(n)} \qquad \longrightarrow \qquad \tilde{\boldsymbol{\mathcal{A}}}_{\boldsymbol{\lambda}}^{(n)} = \boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(n)} - \boldsymbol{\nabla}\chi_{\boldsymbol{\lambda}}^{(n)}$$

Berry courbature $\boldsymbol{\Omega}_{\boldsymbol{\lambda}}^{(n)} = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}^{(n)}$ is invariant:

$$\boldsymbol{\Omega}_{\boldsymbol{\lambda}}^{(n)} \qquad \longrightarrow \qquad \tilde{\boldsymbol{\Omega}}_{\boldsymbol{\lambda}}^{(n)} = \boldsymbol{\Omega}_{\boldsymbol{\lambda}}^{(n)}$$

cf. magnetostatics

$\boldsymbol{\Omega}^{(n)}$ *is a quantity that can be measured*

also true for the geometrical phase $\mathrm{e}^{\mathrm{i}\tilde{\Phi}_{\mathrm{geom}}} = \mathrm{e}^{\mathrm{i}\Phi_{\mathrm{geom}}}$

Two-level system

Hilbert space with dimension 2 : $|\psi\rangle = \alpha|+\rangle \; + \; \beta|-\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\qquad |\alpha|^2 + |\beta|^2 = 1$

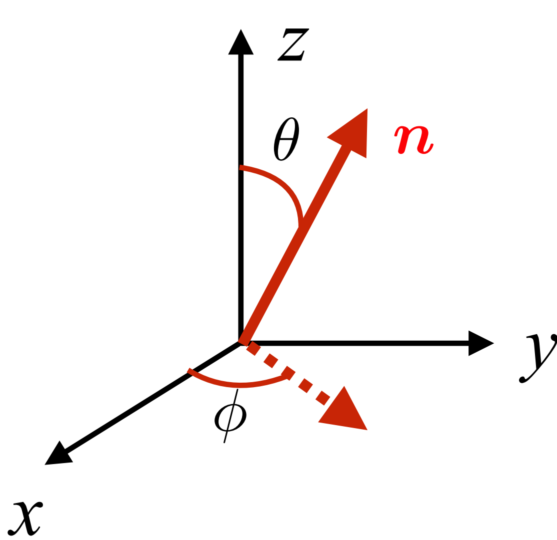
General Hamiltonian : 2x2 Hermitian matrix

$$\hat{H} = E_0 \; \hat{1} \; - \; \boldsymbol{h} \cdot \hat{\boldsymbol{\sigma}}$$

$E_0, \; (h_x, h_y, h_z)$: real numbers

$\hat{\sigma}_j$: Pauli matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$



$\boldsymbol{n} = \dfrac{\boldsymbol{h}}{|\boldsymbol{h}|}$: Unit vector with pointing in the direction (θ, ϕ)

$$\boldsymbol{n} = \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix}$$

$$\hat{H} = E_0 \; \hat{1} \; - \; |\boldsymbol{h}| \; \begin{pmatrix} \cos\theta & \mathrm{e}^{-\mathrm{i}\phi} \sin\theta \\ \mathrm{e}^{\mathrm{i}\phi} \sin\theta & -\cos\theta \end{pmatrix}$$

Bloch sphere and adiabatic following

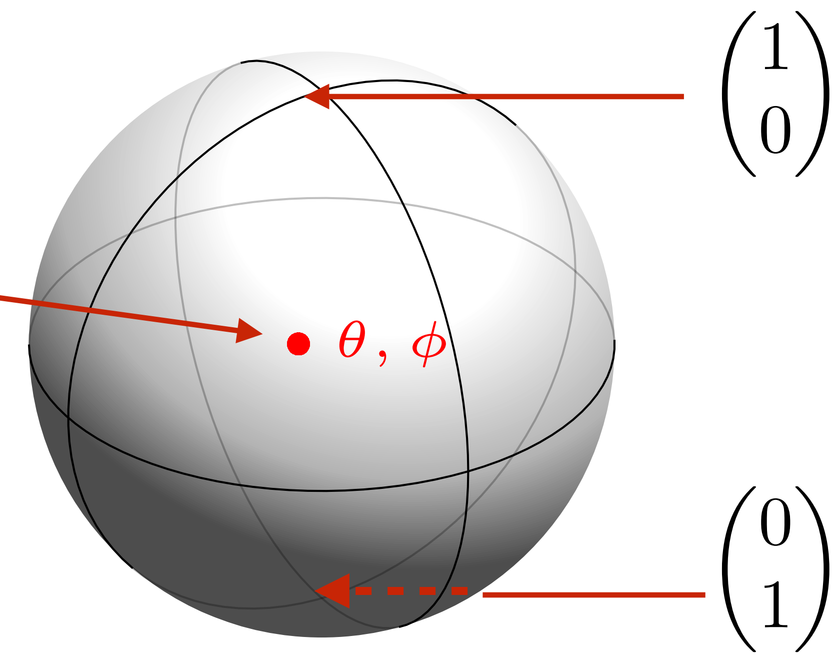
$$\hat{H} = E_0\, \hat{1} \;-\; |\boldsymbol{h}| \begin{pmatrix} \cos\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

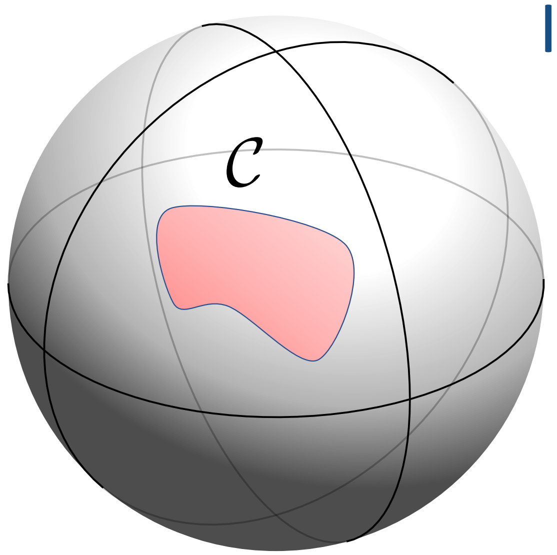
Energies: $E^{(\pm)} = E_0 \pm |\boldsymbol{h}|$

Eigenstates:

$$|\psi^{(-)}\rangle = \begin{pmatrix} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2) \end{pmatrix}$$

$$|\psi^{(+)}\rangle = \begin{pmatrix} \sin(\theta/2) \\ -\mathrm{e}^{\mathrm{i}\phi}\cos(\theta/2) \end{pmatrix}$$





If $\boldsymbol{h}$ varies slowly in time and travels on a closed contour $\mathcal{C}$:

$$\Phi_{\mathrm{geom.}}(\mathcal{C}) = \pm\frac{\alpha}{2}$$

If the trajectory is along the equator:

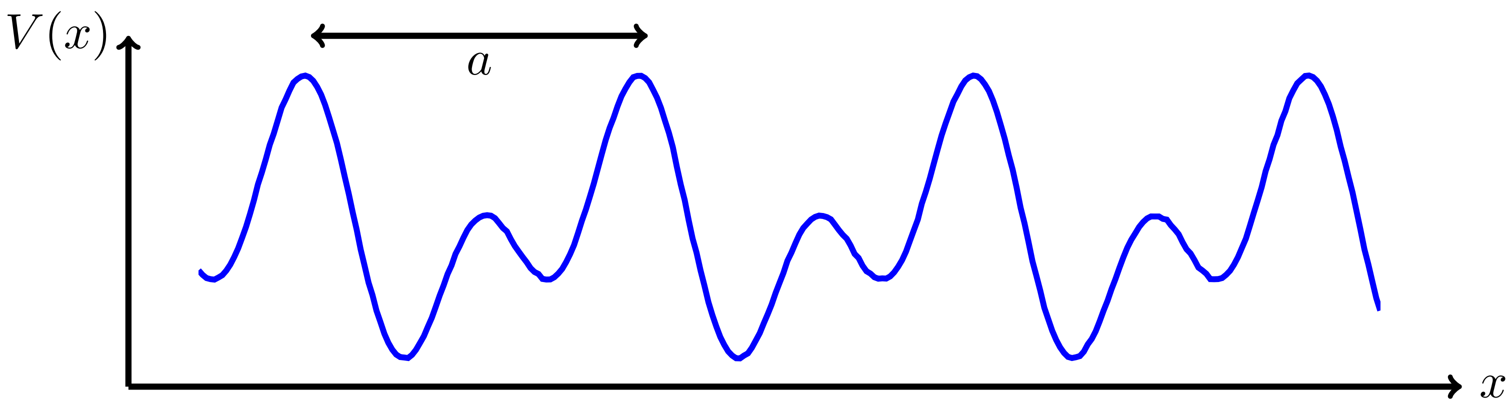
$$\alpha = 2\pi \longrightarrow \Phi_{\mathrm{geom.}}(\mathcal{C}) = \pm\pi$$

2.

Periodic potential, Zak phase and SSH model

Su, Schrieffer & Heeger
Solitons in Polyacetylene
Phys. Rev. Lett. 42, 1698 (1979)

Bloch theorem



$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

$$\hat{p} = -\mathrm{i}\hbar\,\partial_x$$

One can look for the energy eigenstates in terms of Bloch functions

$$\psi_q(x) \;=\; \mathrm{e}^{\mathrm{i}qx}\; u_q(x)$$

Plane wave with momentum ***q***

Periodic function $u_q(x+a) = u_q(x)$

For each ***q***: $\hat{H}_q\; u_q(x) = E_q\; u_q(x)$ $\qquad \hat{H}_q = \frac{\hbar^2}{2m}\left(-\mathrm{i}\partial_x + q\right)^2 + V(x)$

Example of a Hamiltonian depending on an external parameter (here ***q***)

Brillouin zone

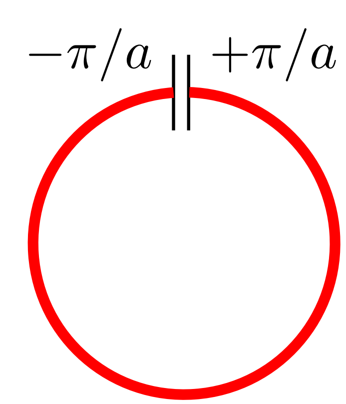
The momentum q can take any value but all values are not all independent

Consider q and $q + \dfrac{2\pi}{a}$

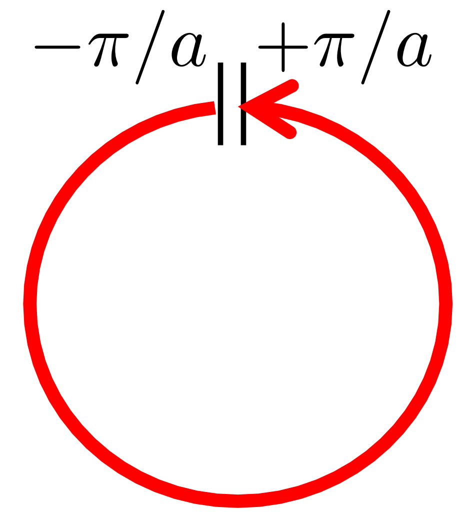
$$\psi_q(x) = \mathrm{e}^{\mathrm{i}qx}\, u_q(x)$$

$$\begin{aligned}\psi_{q+2\pi/a}(x) &= \mathrm{e}^{\mathrm{i}(q+2\pi/a)x}\, u_{q+2\pi/a}(x) \\ &= \mathrm{e}^{\mathrm{i}qx}\, \Big(\underbrace{\mathrm{e}^{\mathrm{i}2\pi x/a}\, u_{q+2\pi/a}(x)}_{\text{periodic function}}\Big)\end{aligned}$$

To avoid double counting, one should reduce the domain of ***q*** to a Brillouin zone



Zak phase



The one-way path from $-\pi/a$ to $+\pi/a$ corresponds to a closed loop in parameter space.

Corresponding to Berry's phase: $\Phi_{\mathrm{geom}} = \int_{-\pi/a}^{+\pi/a} \mathrm{i} \langle u_q | \partial_q u_q \rangle \, \mathrm{d}q$

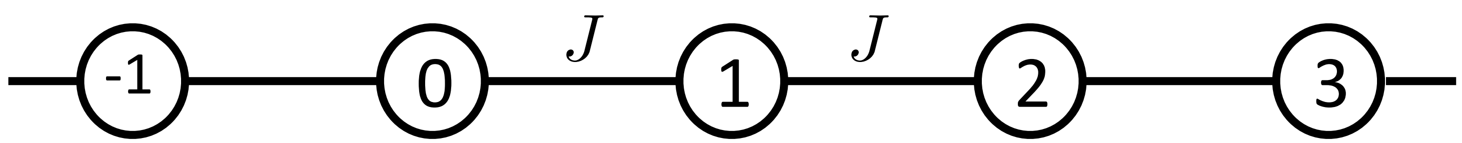
« we have here, for the first time, the appearance of Berry's phase for a one-dimensional parameter space »

Zak, 1989

Requires a non-zero imaginary part for $|u_q\rangle$

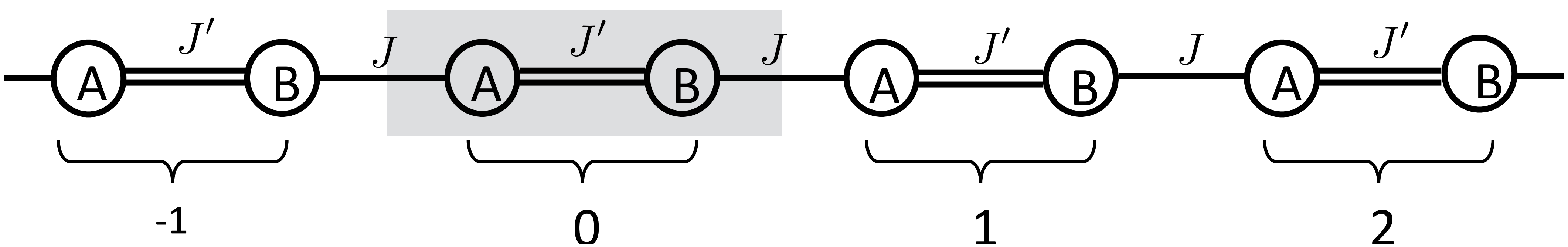
Simplest topological systems in 1D: Tight-binding models

Hubbard model: all sites are equivalent, only nearest neighbor couplings



All states can be taken real: no relevant geometrical phase

Next: SSH model, all equivalent sites with two values for NN couplings

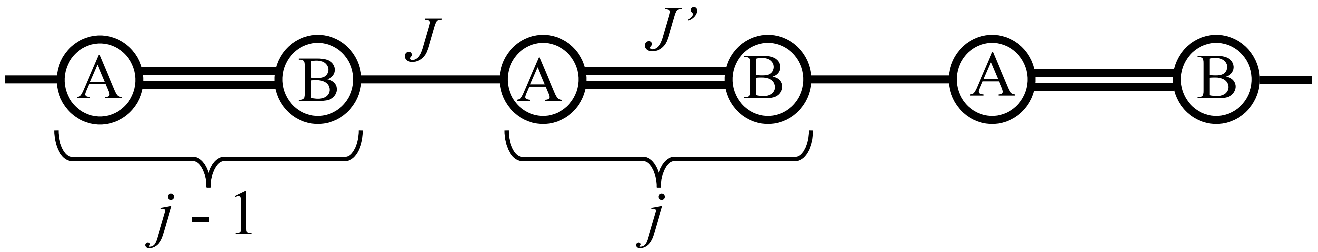


Unit cell with two sites

Periodic function: $|u_q\rangle = \alpha_q \left(\sum_j |A_j\rangle \right) + \beta_q \left(\sum_j |B_j\rangle \right) = \begin{pmatrix} \alpha_q \\ \beta_q \end{pmatrix}$

Pseudo-spin 1/2 for each value of the momentum ***q***

The periodic Hamiltonian for SSH



$$|u_q\rangle = \alpha_q \left(\sum_j |A_j\rangle \right) + \beta_q \left(\sum_j |B_j\rangle \right)$$

Corresponding Bloch function: $$|\psi_q\rangle = \sum_j \mathrm{e}^{\mathrm{i}\, jqa} \left(\alpha_q |A_j\rangle + \beta_q |B_j\rangle \right)$$

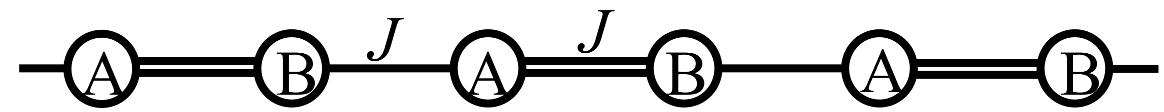
Solution of $\hat{H}|\psi_q\rangle = E_q\, |\psi_q\rangle$ with

$$\hat{H} = -J' \sum_j |B_j\rangle\langle A_j| - J \sum_j |B_{j-1}\rangle\langle A_j| \;+\; \text{h.c.}$$

For each ***q***, one obtains a 2x2 system for the coefficients α_q, β_q:

$$\hat{H}_q \begin{pmatrix} \alpha_q \\ \beta_q \end{pmatrix} = E_q \begin{pmatrix} \alpha_q \\ \beta_q \end{pmatrix}$$ with $$\hat{H}_q = - \begin{pmatrix} 0 & J' + J\,\mathrm{e}^{-\mathrm{i}qa} \\ J' + J\,\mathrm{e}^{\mathrm{i}qa} & 0 \end{pmatrix}$$

Energy spectrum for the SSH model



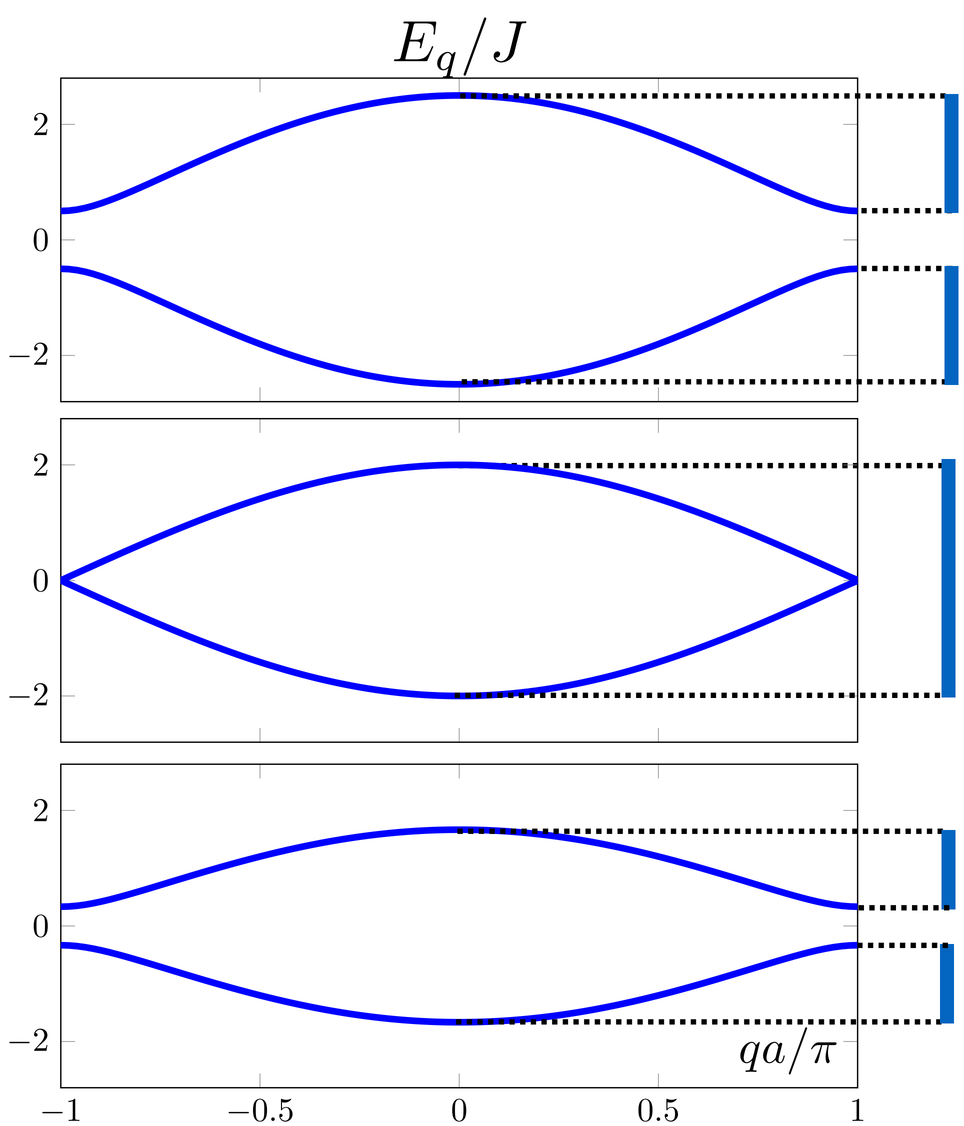
$$\hat{H}_q = -\begin{pmatrix} 0 & J' + J\,\mathrm{e}^{-\mathrm{i}qa} \\ J' + J\,\mathrm{e}^{\mathrm{i}qa} & 0 \end{pmatrix} = -\boldsymbol{h}(q)\cdot\hat{\boldsymbol{\sigma}} \qquad \boldsymbol{h}(q) = \begin{pmatrix} J' + J\cos(qa) \\ J\sin(qa) \\ 0 \end{pmatrix}$$

Energies: $E_q^{(\pm)} = \pm|J' + J\,\mathrm{e}^{\mathrm{i}qa}|$

$J' = \frac{3}{2}J$

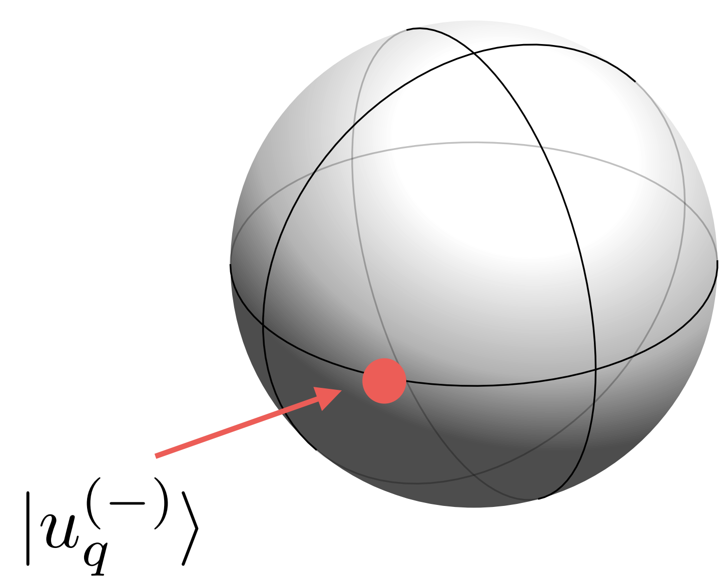
$J' = J$

$J' = \frac{2}{3}J$

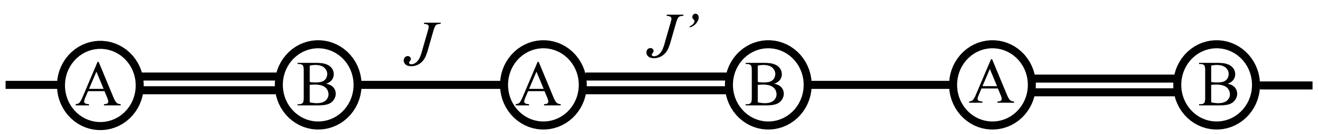


Eigenstates

$h_z(q) = 0$: remains on the equator of the Bloch sphere



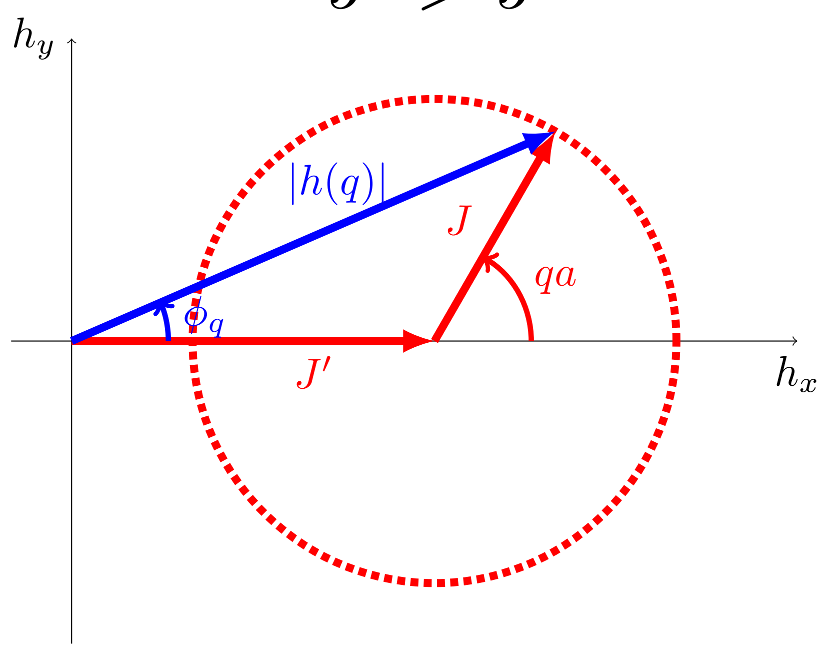
Eigenstates and topology of the SSH model



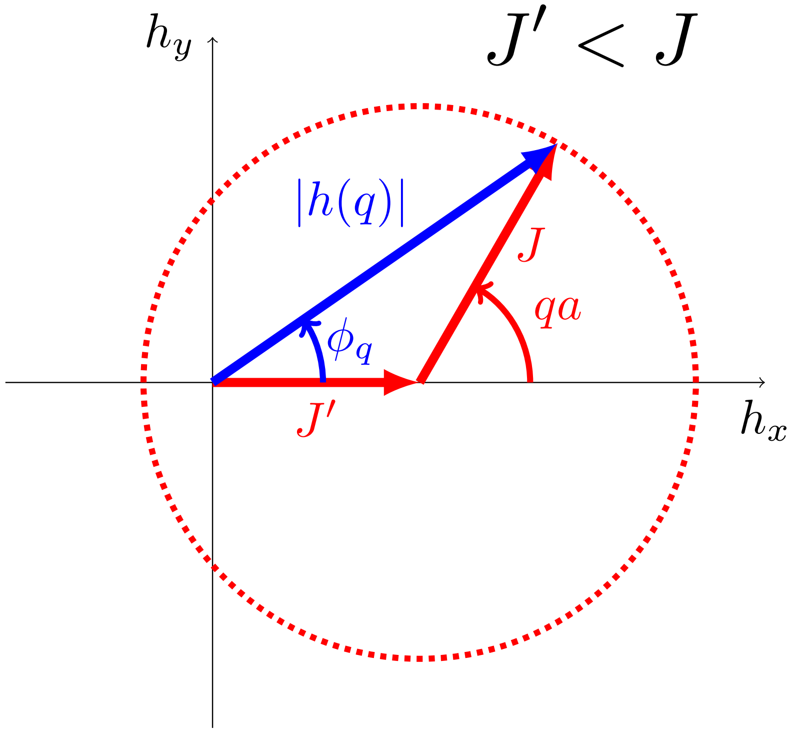
$$\hat{H}_q = -\begin{pmatrix} 0 & J' + J\,\mathrm{e}^{-\mathrm{i}qa} \\ J' + J\,\mathrm{e}^{\mathrm{i}qa} & 0 \end{pmatrix} \qquad |u_q^{(\pm)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp\mathrm{e}^{\mathrm{i}\phi_q} \end{pmatrix}$$

where the phase ϕ_q is defined by $$\mathrm{e}^{\mathrm{i}\phi_q} = \frac{J' + J\,\mathrm{e}^{\mathrm{i}qa}}{|J' + J\,\mathrm{e}^{\mathrm{i}qa}|}$$

$J' > J$

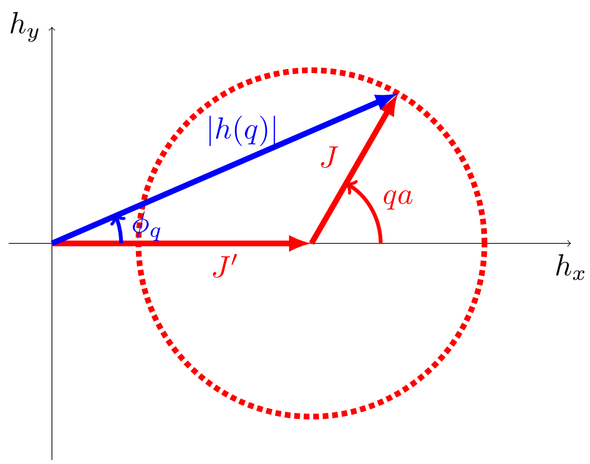


$J' < J$

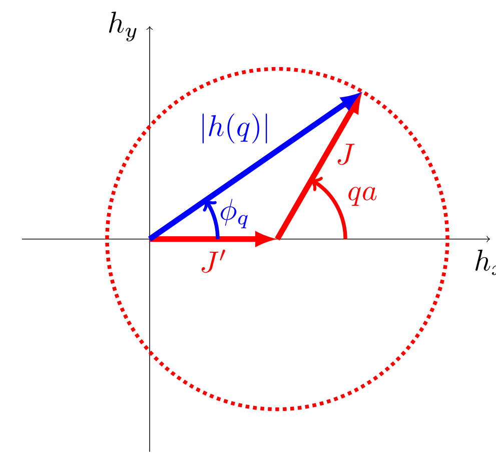


Berry-Zak phase

Trajectoire of $|u_q^{(-)}\rangle$ on Bloch sphere when ***q*** scans the Brillouin zone

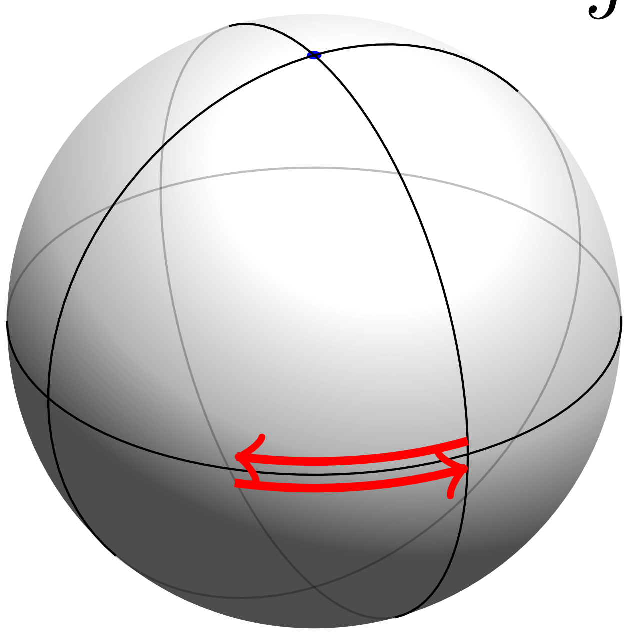


$$|u_q^{(-)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\phi_q} \end{pmatrix}$$

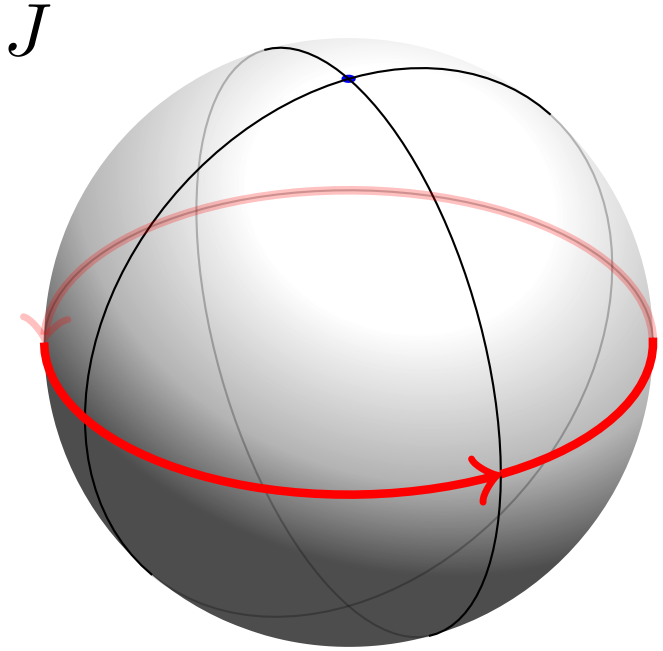


$J' > J$

$J' < J$



$\Phi_{\mathrm{Zak}} = 0$

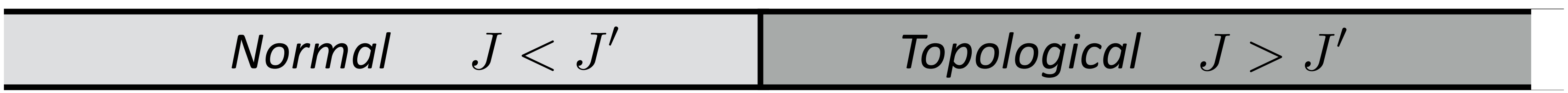


$\Phi_{\mathrm{Zak}} = \pi$

The winding number $N = \Phi_{\mathrm{Zak}}/\pi$ is a topological invariant

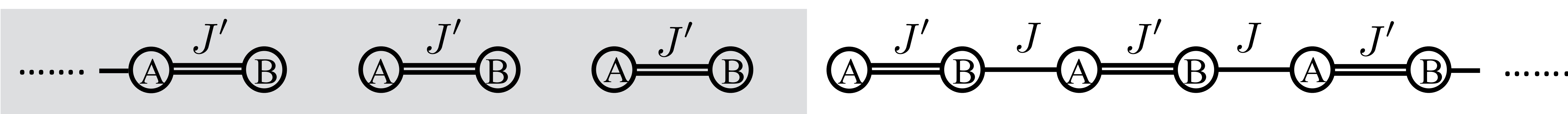
Revealing topology with edge states

One can (relatively easily) show that a localized state will exist at any junction like



even if the junction is soft (J and J' vary slowly in space)

This also applies to a semi-infinite chain:



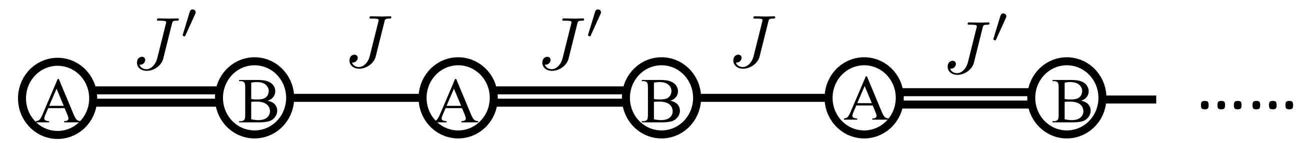
left: $J = 0 \quad < J'$

normal chain

right: $J < J' \quad \text{ou} \quad J > J'$

normal or topological

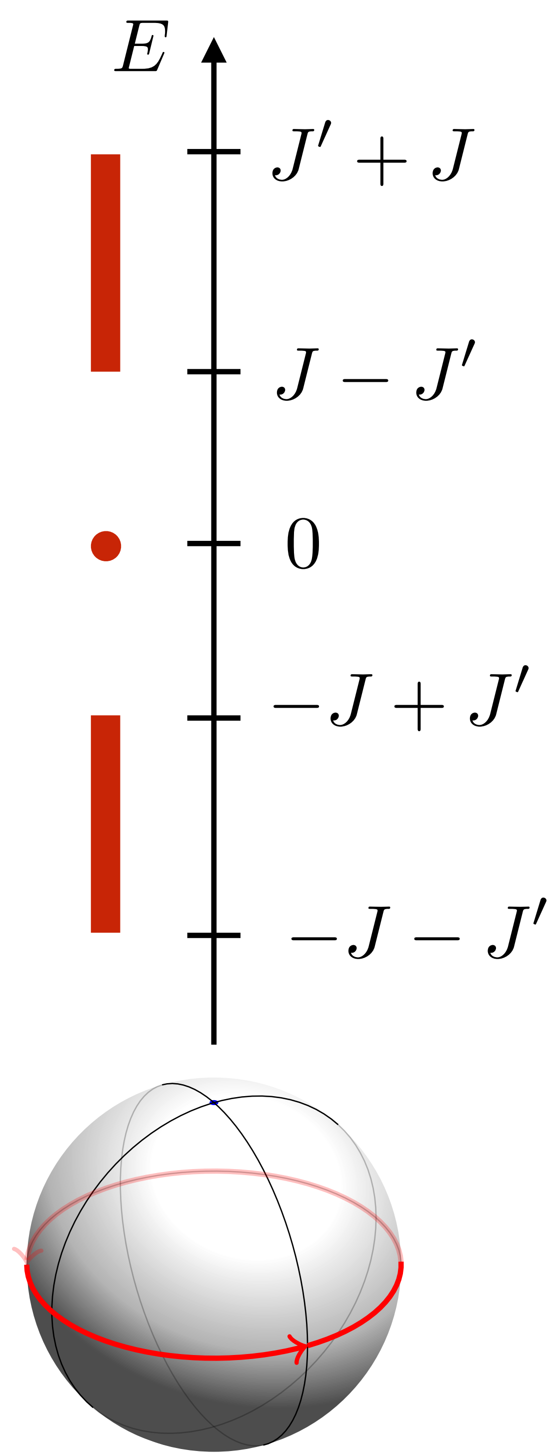
Semi-infinite chain

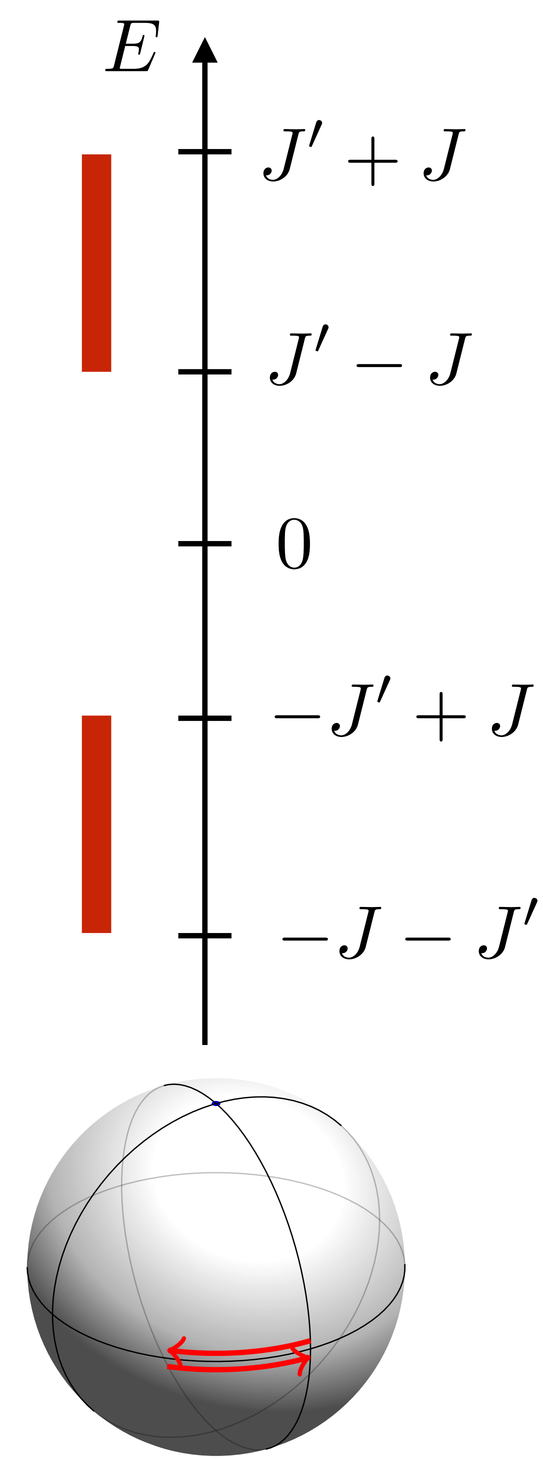


Topological case $J' < J$

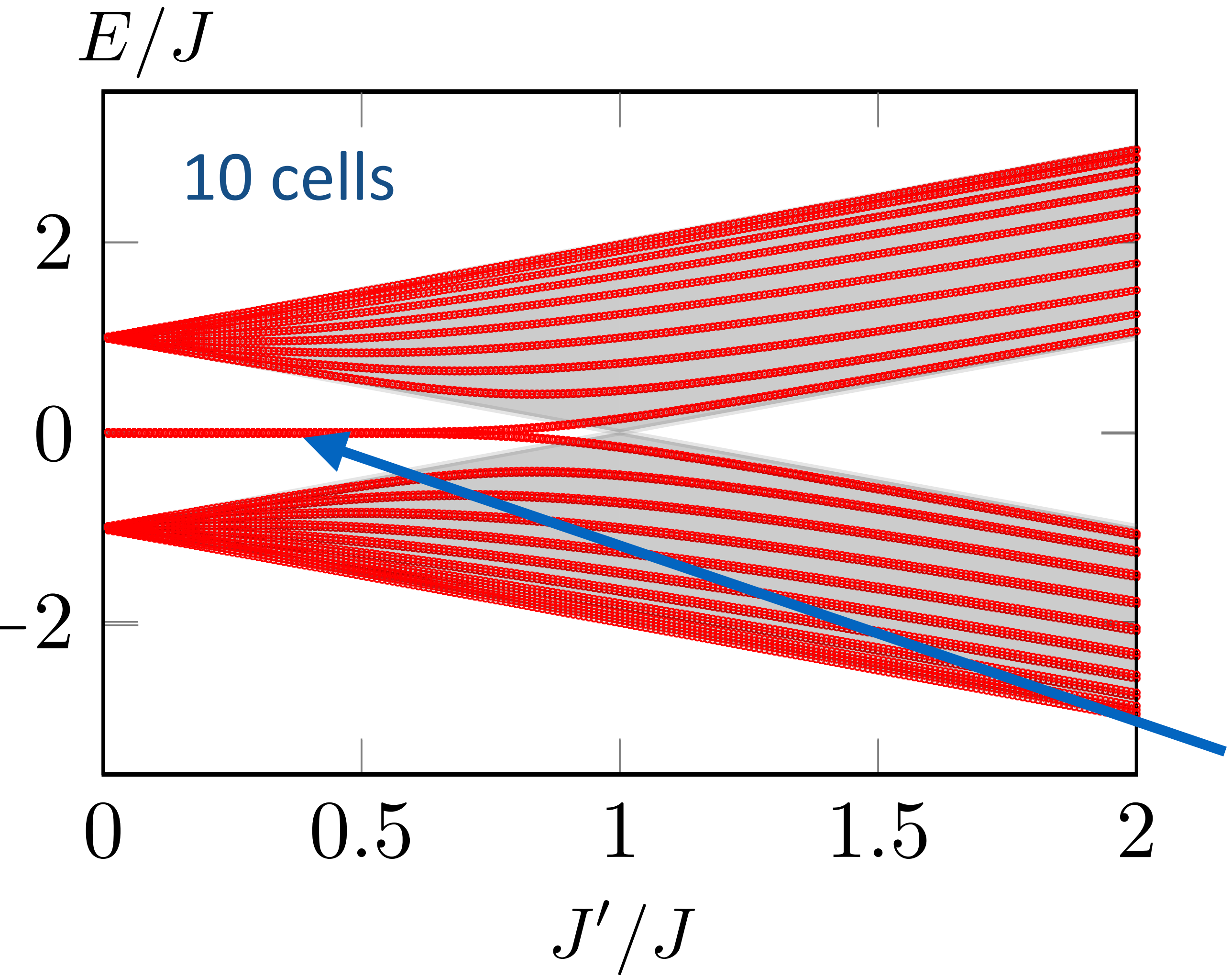
Normal case $J' > J$

Edge state





A finite SSH chain



- We recover the bands of the infinite chain
- Symmetric spectrum: Why?

  Hint: Sublattice symmetry

$$\hat{P}_A \hat{H} \hat{P}_A = 0 \qquad \hat{P}_B \hat{H} \hat{P}_B = 0$$

- In the topological region $J' < J$
  edge states with a quasi-zero energy